Key Concepts: Compound Inequalities
Absolute Value Equations and Inequalities

Intersections and unions

Suppose that **A** and **B** are two sets of numbers.

The intersection of **A** and **B** is the set of all numbers that are in *both* **A** and **B**. This set is symbolized as $A \cap B$.

The union of **A** and **B** is the set of all numbers that are in *either* **A** and/or **B**. This set is symbolized as $A \cup B$.

Example 1

Suppose that **A** is the set $\{1, 2, 3, 4, 5, 6\}$ and **B** is the set $\{1, 3, 5, 7, 9, 11\}$.

a. What are $A \cup B$ and $A \cap B$?

$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9, 11\}$

$A \cap B = \{1, 3, 5\}$

b. What are $A \cup \varnothing$, $A \cap \varnothing$, $A \cup \mathbb{R}$, and $A \cap \mathbb{R}$?

$A \cup \phi = A \cup \{\ \}$
$= \{1, 2, 3, 4, 5, 6\}$

$A \cap \phi = A \cap \{\ \}$
$= \phi$

$A \cup \mathbb{R} = \mathbb{R}$

$A \cap \mathbb{R} = A$
$= \{1, 2, 3, 4, 5, 6\}$

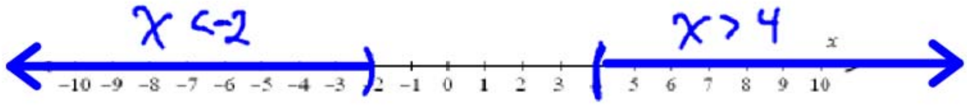
Recall

$\varnothing$ is a symbol used for the empty set – the set with no elements. This set is also denoted by the symbol $\{\ \}$.

$\mathbb{R}$ is a symbol used for the set of all real numbers. This set is also denoted (using interval notation) as $(-\infty, \infty)$.

Example 2

Graph the set $(-\infty,-2)\cup(4,\infty)$ and write a compound inequality whose solution set matches the graph.

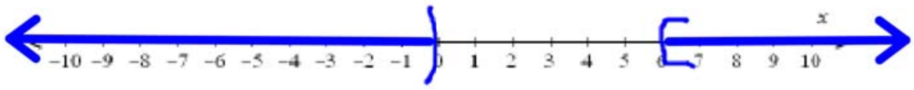


The entire set is the compound inequality

$x<-2$ OR $x>4$

Example 3

Graph the solution set to the compound inequality $x \ge 6$ *or* $x<0$. State the solution set using interval notation.

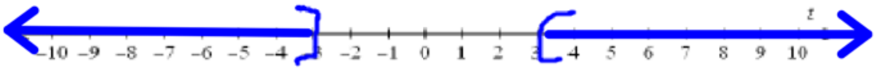


$(-\infty, 0) \quad \cup \quad [6,\infty)$

The solution set is $(-\infty,0)\cup[6,\infty)$

Example 4

Graph the solution set to the compound inequality $3t+1\ge 10$ *or* $5-3t\ge 14$. State the solution set using interval notation.



$3t+1 \ge 10$ or $5-3t \ge 14$

$3t \ge 9$ or $-3t \ge 9$

$\frac{3t}{3} \ge \frac{9}{3}$ OR $\frac{-3t}{-3} \le \frac{9}{-3}$

$t \ge 3$ or $t \le -3$

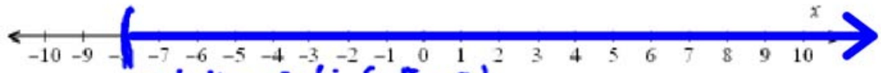
The solution set is

$(-\infty,-3]\cup[3,\infty)$

Example 5

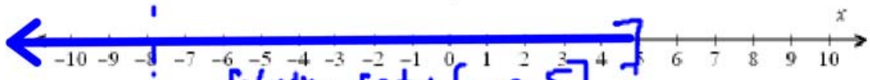
Graph the solution set to each inequality and state the solution set to the compound inequality using interval notation.

$x > -8$



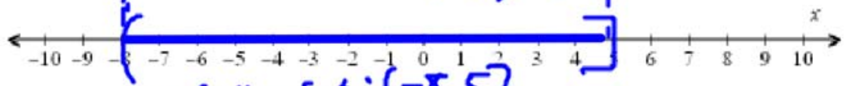
Solution set: $(-8, \infty)$

$x \le 5$



Solution set: $(-\infty, 5]$

Newsflash $x > -8$ *and* $x \le 5$



Solution set: $(-8, 5]$

These "between" inequalities can be written in three pieces: $-8 < x \le 5$

Solutions to "*and*" inequalities

The solution set to an "*and*" inequality between two linear inequalities is *always* one of three things: the empty set, the set of all real numbers, or an interval *between* two numbers (which may or may not include the endpoints).

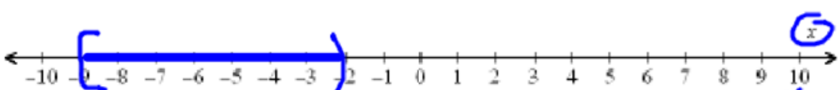
For this reason, "*and*" inequalities are often written as "*between*" inequalities. For example, we could write $x > 3$ *and* $x < 7$ as $3 < x < 7$.

When writing inequalities where the variable is written between two numbers, the two inequality signs *must always open in the same direction*.

$a < x > b$ ← WRONG!! → $a > x < b$

Example 6

Graph the interval $[-9, -2)$ and then write an "*and*" inequality and a "*between*" inequality whose solution sets graph to that interval.



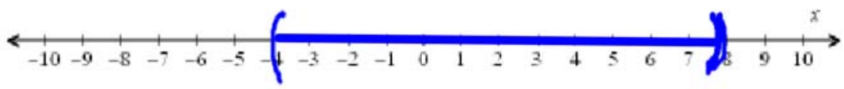
$-9 \le x < -2$

alternatively: $x \ge -9$ and $x < -2$

"and" inequalities sometimes can be written as "between" inequalities

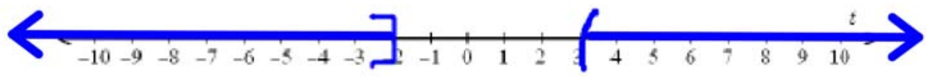
Your turn … work each of the following problems with one or two partners.

1. Graph the solution set to the compound inequality $x > -4$ *and* $x < 8$. State the solution set using interval notation.



The solution set is $(-4, 8)$.

2. Find and graph the solution set to the compound inequality $2-4t \geq 10$ *or* $4+2t > 10$. State the solution set using interval notation.



$2-4t \geq 10$ OR $4+2t > 10$

$-4t \geq 8$ OR $2t > 6$

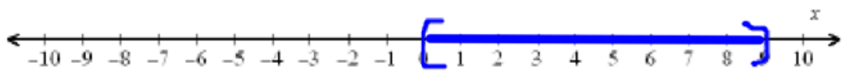
$\frac{-4t}{-4} \leq \frac{8}{-4}$ OR $\frac{2t}{2} > \frac{6}{2}$

$t \leq -2$ OR $t > 3$

The solution set is

$(-\infty, -2] \cup (3, \infty)$.

3. Find and graph the solution set to the compound inequality $-5 \leq 4 - x \leq 4$. State the solution set using interval notion. *Also*, write a compound inequality using the word *and* that has the same solution set.



$-5 \leq 4 - x \leq 4$

$-5 - 4 \leq 4 - x - 4 \leq 4 - 4$

$-9 \leq -x \leq 0$

$\frac{-9}{-1} \geq \frac{-x}{-1} \geq \frac{0}{-1}$

$9 \geq x \geq 0$

The solution set is

$[0, 9]$.

The compound inequality is

$x \geq 0$ and $x \leq 9$

4. Writing $3 \leq x \geq 7$ is bad, m'kay? Rewrite that expression as a compound inequality using the word *and* and discuss what's bad about the situation.

$x \geq 7$ and $x \geq 3$ is not a between inequality because if $x > 7$ it sure shootin isn't between 3 and 7! M'Kay.

Absolute value equations and inequalities

Suppose that k is a *positive* real number.

The equation $|y| = k$ is equivalent to the compound equation $y = k$ *or* $y = -k$.

The inequality $|y| < k$ is equivalent to the compound inequality $-k < y < k$.

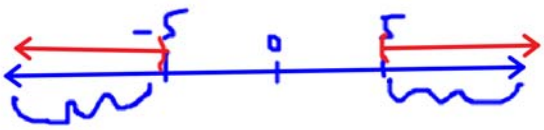
The inequality $|y| > k$ is equivalent to the compound inequality $y > k$ *or* $y < -k$.

Example 7

Rewrite each equation or inequality as a compound equation or inequality.

$|2x+1| \geq 5$

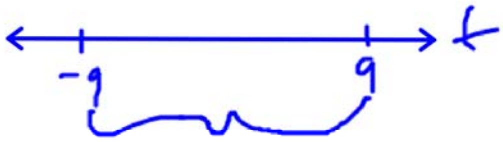
$|2x+1| \geq 5$



$2x+1 \geq 5$ OR $2x+1 \leq -5$

$|t+5| < 9$

$|t+5| < 9$



$-9 < t+5 < 9$

$|3-5x| \leq 9$

$|3-5x| \leq 9$

$-9 \leq 3-5x \leq 9$

$|2-t| + 3 = 8$

$|2-t| + 3 = 8$

$|2-t| = 5$

$2 - t = -5$ OR $2-t=5$

$-2|x| - 5 < 7$

$-2|x| - 5 < 7$

$-2|x| < 12$

$\dfrac{-2|x|}{-2} > \dfrac{12}{-2}$

$|x| > -6$

x is any real number

all absolute values are greater than -6.

Example 8

Find the solution set to each equation or inequality after first rewriting the inequality as a compound inequality. *Begin by completely isolating the absolute value expression.*

$45-9|x-4|=0$

$45-9|x-4|=0$

$45-9|x-4|+9|x-4|=0+9|x-4|$

$45=9|x-4|$

$5=|x-4|$

$x-4=-5$ or $x-4=5$

$x=-1$ or $x=9$

The solution set is $\{-1,9\}$

$|3-2x|+1<6$

$|3-2x|+1<6$

$|3-2x|<5$

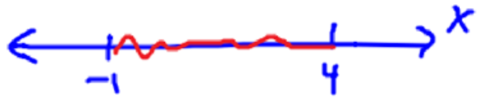
$-5<3-2x<5$

$-5-3<3-2x-3<5-3$

$-8<-2x<2$

$\frac{-8}{-2}>\frac{-2x}{-2}>\frac{2}{-2}$

$4>x>-1$



The solution set is $(-1,4)$

$2|x+6|-6\geq-3$

$2|x+6|-6\geq-3$

$2|x+6|\geq 3$

$|x+6|\geq\frac{3}{2}$

$x+6\geq\frac{3}{2}$ or $x+6\leq-\frac{3}{2}$

$x\geq-\frac{9}{2}$ or $x\leq\frac{-15}{2}$

The solution set is $(-\infty,-\frac{15}{2}]\cup[-\frac{9}{2},\infty)$.

Example 9

Suppose that $f(x)=2x+1$ and $g(x)=3-x$. Find the solution set to each of the following inequalities.

$-2|g(x)|>4$

$-2|g(x)|>4$

$-2|3-x|>4$

$\frac{-2|3-x|}{-2}<\frac{4}{-2}$

$|3-x|<-2$

The solution set is $\varnothing$.

$|f(x)|<0$

$|f(x)|<0$

The solution set is the empty set

$|g(x)|\geq -5$

$|g(x)|\geq -5$

The solution set is $\mathbb{R}$

$-2|f(x+3)|+3\leq 11$

$-2|f(x+3)|+3\leq 11$

$-2|2(x+3)+1|+3\leq 11$

$-2|2x+7|+3\leq 11$

$-2|2x+7|+3-3\leq 11-3$

$-2|2x+7|\leq 8$

$\frac{-2|2x+7|}{-2}\geq\frac{8}{-2}$

$|2x+7|\geq -4$

The solution set is $\mathbb{R}$.

Your turn … work each of the following problems with one or two partners.

1. Find the solution set to $|x| > 7$. Write the solution set using interval notation.

2. Find the solution set to $|x| - 10 < -4$. Write the solution set using interval notation.

3. Find the solution set to $|x - 10| < -4$.

4. Find the solution set to $|f(t)| \geq 8$ where $f(t) = 3 - 5t$. Write the solution set using interval notation.

5. Find the solution set to $|f(4 - x)| = 4$ where $f(x) = 2x - 7$.

6. Find the solution set to $-3|g(x)| + 2 < -10$ where $g(x) = 2x + 4$.