Key Concepts: Graphing form for quadratic equations
Completing the Square

Introductory Activity

Graph each of the following equations on your calculator and enter the requested information.

Equation	Vertex	Number of x-intercepts	Opens Up or Down?
$y=x^2$	(0,0)	1	UP
$y=-x^2$	(0,0)	1	Down
$y=(x-2)^2$	(2,0)	1	UP
$y=(x+3)^2$	(-3,0)	1	UP
$y=x^2-2$	(0,-2)	2	UP
$y=x^2+3$	(0,3)	0	UP
$y=2(x-3)^2-5$	(3,-5)	2	UP
$y=-2(x-3)^2-5$	(3,-5)	0	Down
$y=-\frac{1}{2}(x+2)^2+3$	(-2,3)	2	Down
$y=\frac{1}{2}(x+2)^2+3$	(-2,3)	0	UP

Discussion Activity

Look for patterns in the table. What's the difference between the equations that result in upward opening parabolas vs. downward opening parabolas? What determines the vertex? Can you think of two different ways of determining the number of x-intercepts?

The graphing form for a quadratic equation

The equation $y = a(x-h)^2 + k$ (where $a \neq 0$) graphs to a parabola whose vertex is the point (h, k) and whose axis of symmetry is the vertical line $x = h$.

Furthermore, if $a > 0$ the parabola opens upwards and if $a < 0$ the parabola opens downward.

Examples

Determine, without graphing, the vertex, axis of symmetry, opening orientation, and number of x-intercepts for each of the following parabolas.

Equation	Vertex/Axis	Number of x-intercepts	Opens Up or Down?
$y = -7(x-6)^2 + 8$	V: $(6, 8)$ A: $x = 6$	2	Down
$y = x^2 + 4$	V: $(0, 4)$ A: $x = 0$	0	Up
$y = x^2 + 10x - 6$ $y = (x+5)^2 - 31$	$(-5, -31)$	2	Up

$$y = x^2 + 10x - 6 \qquad \left[\tfrac{1}{2}(10)\right]^2$$

$$y = (x^2 + 10x + \underline{25}) - 6 - \underline{25}$$

$$\underbrace{y = (x+5)^2 - 31}_{\text{graphing form}}$$

Completing the square when the leading coefficient is 1 ($y = x^2 \pm bx + c$)

$$y = x^2 \pm bx + c$$
$$y = \left(x^2 \pm bx\right) + c$$
$$y = \left(x^2 \pm bx + \left(\frac{b}{2}\right)^2\right) + c - \left(\frac{b}{2}\right)^2$$

1. Group the square and linear terms and add the square of half the linear coefficient inside the parentheses; balance this action outside the parentheses

$$y = \left(x \pm \frac{b}{2}\right)^2 + \left(c - \left(\frac{b}{2}\right)^2\right)$$

2. Factor the prefect square trinomial. The plus/minus sign should agree with the sign on the line above.

Examples

State the vertex of each parabola after first completing the square.

$y = x^2 - 16x + 12$

$\left[\frac{1}{2}(16)\right]^2$

$y = (x^2 - 16x + \underline{64}) + 12 - \underline{64}$

$y = (x-8)^2 - 52$ Vertex: $(8, -52)$

$y = x^2 + 20x$

$y = (x^2 + 20x + \underline{100}) - \underline{100}$

$y = (x+10)^2 - 100$ Vertex: $(-10, -100)$

$y = x^2 + 3x + \frac{3}{4}$

$y = (x^2 + 3x + \underline{\frac{9}{4}}) + \frac{3}{4} - \underline{\frac{9}{4}}$

$y = \left(x + \frac{3}{2}\right)^2 - \frac{3}{2}$

vertex: $\left(-\frac{3}{2}, -\frac{3}{2}\right)$

Completing the square when the leading coefficient is not 1 ($y = ax^2 + bx + c$)

$y = -2x^2 + 8x + 3$
$y = -2\left(x^2 - 4x\right) + 3$

1. Factor the squared-term coefficient from the squared term and the linear term.

$y = -2\left(x^2 - 4x + 4\right) + 3 + 8$

2. Add the square of half the ***new*** linear coefficient inside the parentheses and balance this action outside the parentheses

$y = -2(x-2)^2 + 11$

3. Factor the prefect square trinomial.

Examples

State the vertex of each parabola after first completing the square.

$y = 4x^2 + 24x - 20$

$y = 4x^2 + 24x - 20$

$y = 4(x^2 + 6x + \underline{9}) - 20 - \underline{4(9)}$

$y = 4(x+3)^2 - 56$

Vertex: $(-3, -56)$

$y = -x^2 - 6x - 9$

$y = -x^2 - 6x - 9$

$y = -1(x^2 + 6x + \underline{9}) - 9 - \underline{(-1)(9)}$

$y = -1(x+3)^2 + 0$

$y = -(x+3)^2$

Vertex: $(-3, 0)$

$y = -3x^2 + 6x + 3$

$y = -3x^2 + 6x + 3$

$y = -3(x^2 - 2x + \underline{1}) + 3 - \underline{(-3)(1)}$

$y = -3(x-1)^2 + 6$

The vertex is $(1, 6)$

$y = x^2 + 5x + \frac{13}{4}$

$\left[\frac{1}{2}(5)\right]^2 = \frac{5}{2} \cdot \frac{5}{2} = \frac{25}{4}$

$y = x^2 + 5x + \frac{13}{4}$

$y = (x^2 + 5x + \underline{\frac{25}{4}}) + \frac{13}{4} - \underline{\frac{25}{4}}$

$y = (x + \frac{5}{2})^2 - 3$

The vertex is $(-\frac{5}{2}, -3)$

$y = 9x^2 - 36x + 36$

$y = 9x^2 - 36x + 36$

$y = 9(x^2 - 4x + \underline{4}) + 36 - \underline{9(4)}$

$y = 9(x-2)^2$

The vertex is $(2, 0)$

The parabolas associated with the equations stated as (a)-(d) are each shown in one of figures 1-8. Match the equations to the parabolas. Please note that scales have deliberately been omitted from the graphs so that you cannot answer the question by simply plugging in points. The same scale was used, however, for each of the graphs. <u>Do not graph the parabolas on your calculator; that's not the point!!</u>

a. The parabola $y=(x-2)^2-3$ is shown in Figure ___2___.

Concave up

Vertex: (2, -3) ; happy
(QUAD IV)

b. The parabola $y=-(x+2)^2+3$ is shown in Figure ___1___.

Concave down

(-2, 3) ; sad
QUAD II

c. The parabola $y=(x+3)^2$ is shown in Figure ___5___.

(-3, 0) ; concave up

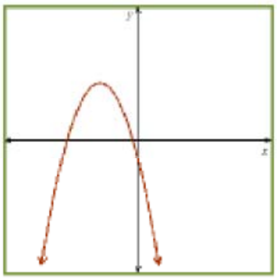


Figure 1

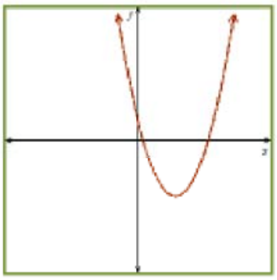


Figure 2

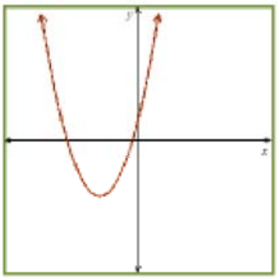


Figure 3

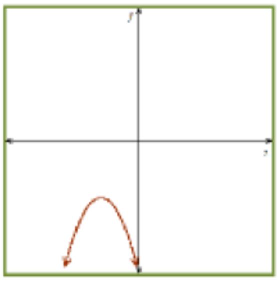


Figure 4

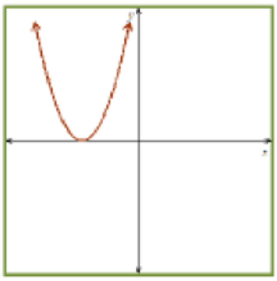


Figure 5

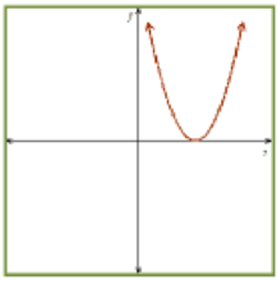


Figure 6

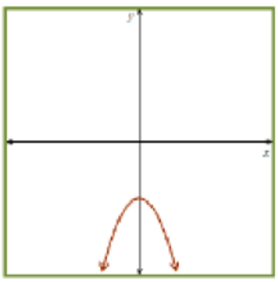


Figure 7

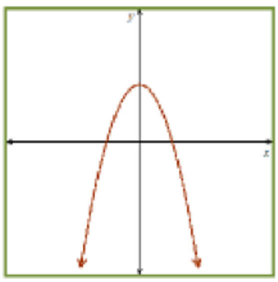


Figure 8

$-1 + 3 = 4a - 3 + 3$

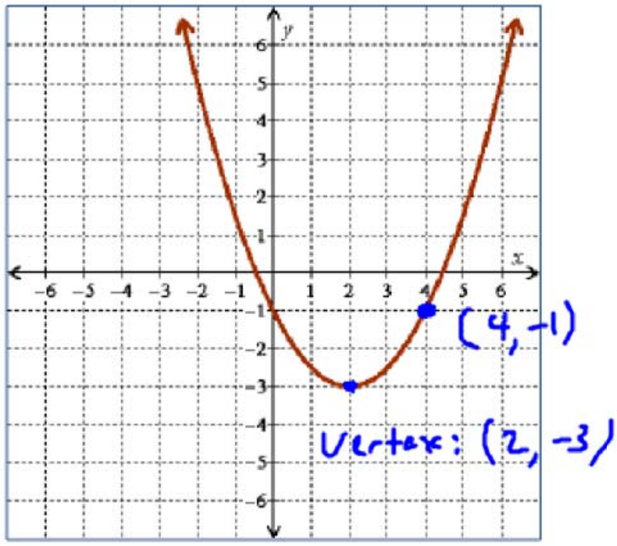
$2 = 4a$

$\frac{2}{4} = \frac{4a}{4}$

$\frac{1}{2} = a$

Examples

Find the equations for each of the following parabolas. Write the equations in standard form.



From the vertex, we know

$y = a(x-2)^2 - 3$

From $(4, -1)$

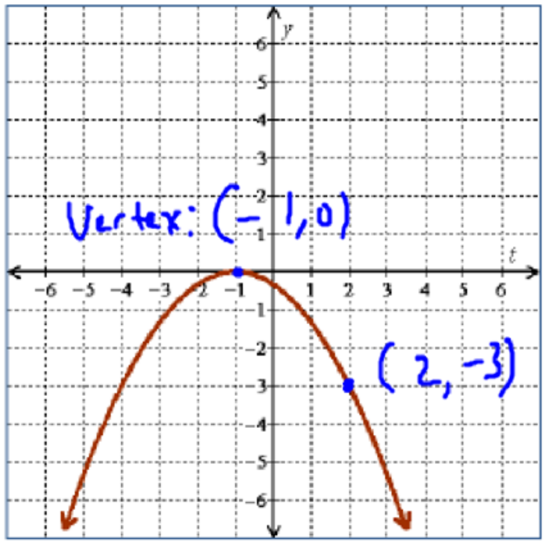
$-1 = a(4-2)^2 - 3$

$-1 = 4a - 3$

$2 = 4a$

$\frac{1}{2} = a$

The parabola is $y = \frac{1}{2}(x-2)^2 - 3$



From the vertex

$y = a(x+1)^2 + 0$

From $(2, -3)$

$-3 = a(2+1)^2$

$-3 = a \cdot 3^2$

$-3 = 9a$

$-\frac{1}{3} = a$

The parabola is $y = -\frac{1}{3}(x+1)^2$

Examples

For each parabola, complete the square and then determine the vertex and x-intercepts of the parabola. Confirm your results on your graphing calculator.

$y = x^2 + 2x - 24$

To find the x-intercepts, $y = 0$

$x^2 + 2x - 24 = 0$

$(x+6)(x-4) = 0$

$x + 6 = 0$ or $x - 4 = 0$

$x = -6$ or $x = 4$

The x-intercepts are $(-6, 0)$ and $(4, 0)$.

$y = x^2 + 2x - 24$

$y = (x^2 + 2x + \underline{1}) - 24 - \underline{1}$

$y = (x+1)^2 - 25$

The vertex is $(-1, -25)$

$y = x^2 - 6x + 2$

$x^2 - 6x + 2 = 0$

$a = 1,\ b = -6,\ c = 2$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(2)}}{2(1)}$

$x = \frac{6 \pm \sqrt{28}}{2}$

$x = \frac{6 \pm \sqrt{4 \cdot 7}}{2}$

$x = \frac{6 \pm \sqrt{4}\sqrt{7}}{2}$

$x = \frac{6 \pm 2\sqrt{7}}{2}$

$x = \frac{6}{2} \pm \frac{2\sqrt{7}}{2}$

$x = 3 \pm \sqrt{7}$

The x-intercepts are $(3+\sqrt{7}, 0)$ and $(3-\sqrt{7}, 0)$

$y = x^2 - 6x + 2$

$y = (x^2 - 6x + \underline{9}) + 2 - \underline{9}$

$y = (x-3)^2 - 7$

The vertex is $(3, -7)$

$y = -2x^2 + 8x - 200$

$-2x^2 + 8x - 200 = 0$

$a = -2,\ b = 8,\ c = -200$

$x = \frac{-8 \pm \sqrt{8^2 - 4(-2)(-200)}}{2(-2)}$

$x = \frac{-8 \pm \sqrt{-1534}}{-4}$ ← not a real number!

There are no x-intercepts!

$y = -2x^2 + 8x - 200$

$y = -2(x^2 - 4x + \underline{4}) - 200 - \underline{(-2)(4)}$

$y = -2(x-2)^2 - 192$

The vertex is $(2, -192)$