Key Concepts: Polynomials
Definitions, Addition and Subtraction, Multiplication

Definitions

A ***Polynomial Term*** is a constant, or a product of variables raised to whole number powers, or a product of a constant and variables raised to whole number powers.

The constant factor of a polynomial term is called the ***coefficient*** of the term.

The sum of the exponents on the variable factors is called the ***degree*** of the term. A constant term has degree 0.

A single polynomial term or the sum of two or more polynomial terms is called a ***polynomial***.

A polynomial with **one term** is called a ***monomial***. A polynomial with **two terms** is called a ***binomial***. A polynomial with **three terms** is called a ***trinomial***.

Example

List the terms of the polynomial $-4x^2y^7 + x^4 - xy^2 - 7y + 9$. Also, state the coefficient and degree of each term.

$+(-xy^2)$

Term	$-4x^2y^7$	x^4	$-xy^2$	$-7y$	9
Coefficient	-4	1	-1	-7	9
Degree	9	4	3	1	0

$-7y$ is a linear term $\quad -x^1y^2 \quad -7y^1$

9 is a constant term

Addition and Subtraction of Polynomials

1. Distribute the subtraction sign (if subtracting). Remove parentheses regardless.
2. Combine the like terms. Like terms have the same variable factors (including exponents).

$x^2 x^2$

Examples

Find each sum or difference.

trinomial binomial

$\left(3x^2+7x-2\right)+\left(-5x^2-7x\right)$

$$
\begin{aligned}
(3x^2+7x-2)+(-5x^2-7x) &= 3x^2+7x-2-5x^2-7x \leftarrow \\
&= (3x^2-5x^2)+(7x-7x)-2 \\
&= -2x^2+0x-2 \\
&= -2x^2-2 \leftarrow
\end{aligned}
$$

$\left(8x^2+2x-7\right)-\left(3x^2-9x+2\right)$

$$
\begin{aligned}
(8x^2+2x-7)-(3x^2-9x+2) &= 8x^2+2x-7-3x^2+9x-2 \leftarrow \\
&= (8x^2-3x^2)+(2x+9x)+(-7-2) \\
&= 5x^2+11x-9 \leftarrow
\end{aligned}
$$

Find $\left(3n^4 - 2n^2 + 4n + 2\right) - \left(-n^3 - 2n^2 - 4n + 2\right)$

$$\left(3n^4 - 2n^2 + 4n + 2\right) - \left(-n^3 - 2n^2 - 4n + 2\right)$$
$$= 3n^4 \cancel{- 2n^2} + 4n \cancel{+ 2} + n^3 \cancel{+ 2n^2} + 4n \cancel{- 2}$$
$$= 3n^4 + n^3 + 8n$$

Multiplying Polynomials

1. Use the distributive property of multiplication to eliminate all grouping symbols.
2. Simplify each polynomial term.
3. Combine the like terms.

Examples

Find each product.

$4x^2\left(3x^3 - 4x^2 + 9x + 11\right)$

$$4x^2(3x^3 - 4x^2 + 9x + 11)$$
$$= (4x^2)(3x^3) - (4x^2)(4x^2) + (4x^2)(9x) + (4x^2)(11)$$
$$= (4\cdot 3)(x^2x^3) - (4\cdot 4)(x^2x^2) + (4\cdot 9)(x^2x) + (4\cdot 11)x^2$$
$$= 12x^5 - 16x^4 + 36x^3 + 44x^2$$

$-4v^2\left(2-v^2+11v-7\right)$

$$-4v^2(2-v^2+11v-7) = -8v^2+4v^4-44v^3+28v^2$$
$$= 4v^4-44v^3+20v^2$$

decreasing degree order
(at least when there's only one variable)

Multiply the polynomials $7x+3$ and x^2-5x+2.

$$(7x+3)(x^2-5x+2)$$
$$= (7x+3)(x^2) - (7x+3)(5x) + (7x+3)(2)$$
$$= (7x)(x^2)+(3)(x^2)-(7x)(5x)-(3)(5x)+(7x)(2)+(3)(2)$$
$$= 7x^3+3x^2-35x^2-15x+14x+6$$
$$= 7x^3-32x^2-x+6$$

Multiply the polynomials $6-x$ and $2x^2-x-2$.

$$(6-x)(2x^2-x-2) = 12x^2-6x-12-2x^3+x^2+2x$$
$$= -2x^3+13x^2-4x-12$$

$$\boxed{(6-x)(2x^2-x-2)}$$

$6x^3$
$7y^3$
$4xy^2$
$5x^2y$

$6x^3+5x^2y+4xy^2+7y^3$

Multiply the polynomials $2x+2y$ and $3x-y+5$.

$$(2x+2y)(3x-y+5) = 6x^2-2xy+10x+6xy-2y^2+10y$$
$$= 6x^2+4xy-2y^2+10x+10y$$

Multiply the polynomials $2-y$ and $-3y^2+7y-8$.

$$(2-y)(-3y^2+7y-8) = -6y^2+14y-16+3y^3-7y^2+8y$$
$$= 3y^3-13y^2+22y-16$$

Find the product $(6x^2+x-4)(-x^2-x+7)$.

$$(6x^2+x-4)(-x^2-x+7)$$
$$= -6x^4-6x^3+42x^2-x^3-x^2+7x+4x^2+4x-28$$
$$= -6x^4-7x^3+45x^2+11x-28$$

Quick sort of check

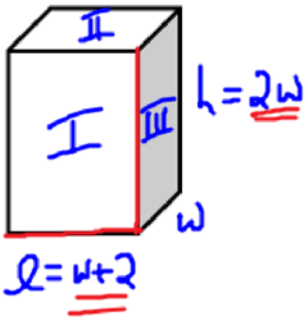
Let $x=1$

$6x^2 = 6\cdot 1^2 = 6\cdot 1 = 6$

original: $(6+1-4)(-1-1+7) = (3)(5) = 15$

Final: $-6-7+45+11-28 = 56-41 = 15$

Find and simplify the volume and surface area of a cube if the height of the cube is twice the width and the length is 2 more than the width. Use the variable w for the width.



$$V = lwh$$
$$= (w+2)\cdot w\cdot 2w$$
$$= (w+2)\cdot 2w^2$$
$$= 2w^3+4w^2$$

$2\mathrm{I}$ $\quad 2\mathrm{II}$ $\quad 2\mathrm{III}$ ← in my head

$$SA = 2\cdot 2w(w+2) + 2\cdot w\cdot(w+2) + 2\cdot 2w\cdot w$$
$$= 4w^2+8w+2w^2+4w+4w^2$$

$$= 10w^2+12w$$

F O I L

$$(a+b)(c+d) = ac + ad + bc + bd$$

$(a+b)(c+d)$	ac	**F**irst terms
$(a+b)(c+d)$	ad	**O**utside terms
$(a+b)(c+d)$	bc	**I**nside terms
$(a+b)(c+d)$	bd	**L**ast terms

Examples

Use FOIL to expand each binomial product and, of course, simplify the result.

$(x+7)(x-2)$

F O I L

$$(x+7)(x-2) = (x)(x) + (x)(-2) + (7)(x) + (7)(-2)$$
$$= x^2 - 2x + 7x - 14$$
$$= x^2 + 5x - 14$$

$(y-9)(y+11)$

F O I L

$$(y-9)(y+11) = (y)(y) + (y)(11) + (-9)(y) + (-9)(11)$$
$$= y^2 + 11y - 9y - 99$$
$$= y^2 + 2y - 99$$

$(x^2-8)(x+3)$

$$(x^2-8)(x+3) = x^3 + 3x^2 - 8x - 24$$

$\left(\frac{2}{3}t+\frac{4}{3}\right)\left(\frac{1}{2}t-\frac{3}{4}\right)$

$$\left(\frac{2}{3}t+\frac{4}{3}\right)\left(\frac{1}{2}t-\frac{3}{4}\right) = \frac{2}{6}t^2 - \frac{6}{12}t + \frac{4}{6}t - 1$$
$$= \frac{1}{3}t^2 - \frac{1}{2}t + \frac{2}{3}t - 1$$
$$= \frac{1}{3}t^2 - \frac{3}{6}t + \frac{4}{6}t - 1$$
$$= \frac{1}{3}t^2 + \frac{1}{6}t - 1$$

The binomials $a+b$ and $a-b$ are called conjugates. For example, $x-2$ and $x+2$ are conjugates. Likewise $y^6 - x^2$ and $y^6 + x^2$ are conjugates.

Let's use FOIL to multiply the conjugate pair $a+b$ and $a-b$ to see what's so darn special about this pair of binomials.

$$(a+b)(a-b) = a^2 - ab + ab - b^2$$
$$= a^2 - b^2$$

Examples

Use the special product for conjugates to find each of the following.

$(x+2)(x-2)$

$$(x+2)(x-2) = x^2 - 2^2$$
$$= x^2 - 4$$

$$\boxed{(a^m)^n = a^{mn}}$$

$\left(y^6 - x^2\right)\left(y^6 + x^2\right)$

$$(y^6 - x^2)(y^6 + x^2) = (y^6)^2 - (x^2)^2$$
$$= y^{12} - x^4$$

$$(ab)^3 = a^3 b^3$$

Q: So what's up with exponents not distributing over addition and subtraction? Like, why doesn't $(a+b)^2 = a^2 + b^2$?

A: Ummmm....., 'cuz it doesn't.

Examples

Find each of the following.

F O I L

$(a+b)^2$

$$(a+b)^2 = (a+b)(a+b)$$
$$= a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$

$$(3+4)^2 = 7^2 = 49$$
$$3^2 + 4^2 = 9 + 16 = 25$$
$$(3+4)^2 \neq 3^2 + 4^2$$

$(a-b)^2$

$$(a-b)^2 = (a-b)(a-b)$$
$$= a^2 - ab - ab + b^2$$
$$= a^2 - 2ab + b^2$$

$(4x+5y)^2$

$$(4x+5y)^2 = (4x+5y)(4x+5y)$$
$$= 16x^2 + 20xy + 20xy + 25y^2$$
$$= 16x^2 + 40xy + 25y^2$$

$$(t-4)(t-4)^2$$
$$(t-4)\ (t-4)(t-4)$$

$(t-4)^3$

$$(t-4)^3 = (t-4)(t-4)(t-4)$$
$$= (t-4)\left[t^2-4t-4t+16\right]$$
$$= (t-4)(t^2-8t+16)$$
$$= t^3-8t^2+16t-4t^2+32t-64$$
$$= t^3-12t^2+48t-64$$

$(2x^2-3x+5)^2$

$$(2x^2-3x+5)^2 = (2x^2-3x+5)(2x^2-3x+5)$$
$$= 4x^4-6x^3+10x^2-6x^3+9x^2-15x$$
$$+10x^2-15x+25$$
$$= 4x^4-12x^3+29x^2-30x+25$$