Option 1

Is the integrand something directly on the antiderivative sheet? Is the integrand really close to something on the antiderivative sheet (via a trivial chain rule you can figure out in reverse)?

Examples: $\int \sin(2x)\,dx = -\frac{1}{2}\cos(2x) + C$ $\qquad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C$

$$\int e^{1+4t}\,dt = \frac{1}{4}e^{1+4t} + C$$

Option 2

Can the integrand be algebraically manipulated into something trivial like the power rule? Can trig identities be used to manipulate the integrand into something on the antiderivative sheet?

Examples:

$$\begin{aligned}\int \left(x^2-4\right)^2 dx &= \int \left(x^4 - 8x^2 + 16\right)dx \\ &= \frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x + C\end{aligned}$$

$$\begin{aligned}\int \frac{3-\sqrt{t}}{t^2}\,dt &= \int \left(3t^{-2} - t^{-3/2}\right)dt \\ &= -3t^{-1} + 2t^{-1/2} + C \\ &= -\frac{3}{t} + \frac{2}{t^{1/2}} + C\end{aligned}$$

$$\begin{aligned}\int \frac{dt}{\sqrt[3]{e^t}} &= \int e^{-t/3}\,dt \\ &= -3e^{-t/3} + C \\ &= -\frac{3}{\sqrt[3]{e^t}}\end{aligned}$$

$$\begin{aligned}\int \tan(\theta)\cos(\theta)\,d\theta &= \int \frac{\sin(\theta)}{\cos(\theta)}\cos(\theta)\,d\theta \\ &= \int \sin(\theta)\,d\theta \\ &= -\cos(\theta) + C\end{aligned}$$

Option 3

Can the integral be resolved via an "obvious" composite function based substitution? That is, does the integrand have form $\sin(u)$, $\cos(u)$, u^n, e^u, etc? To make this determination, write down "u" and determine "du." If the ***<u>exact</u>*** variable expression in du is present in the integrand, then you are good to go. If not, move on to option 4.

Examples:

$$\int x\sin\left(x^2\right)dx$$

Let $u = x^2$. Then $du = 2x\,dx$ so we can replace $x\,dx$ with $\frac{du}{2}$ resulting in the integral

$$\int \frac{1}{2}\sin(u)\,du$$

$$\int \frac{\sec^2(\ln(y))}{y}\,dy$$

Let $u=\ln(y)$. Then $du=\frac{dy}{y}$ so the integral has form $\int \sec^2(u)\,du$

$$\int \frac{3x^2-2}{\sqrt[7]{(x^3-2x)^6}}\,dx$$

Let $u=x^3-2x$. Then $du=(3x^2-2)\,dx$ so the integral has form $\int u^{-6/7}\,du$

Counterexamples:

$$\int x^2\sin(x^2)\,dx$$

If we let $u=x^2$, then $du=2x\,dx$. There is an extra factor x hanging outside the sine factor. Move on to another option.

$$\int y\sec^2(\ln(y))\,dy$$

If we let $u=\ln(y)$, then $du=\frac{dy}{y}$. The factor of y is in the wrong place. Move on to another option.

$$\int \frac{3x^2}{\sqrt[7]{(x^3-2x)^6}}\,dx$$

If we let $u=x^3-2x$, then $du=(3x^2-2)\,dx$. The absence of a required ***term*** is a total deal breaker substitution-wise. Move on to integration by parts and see if you have some luck there.

Option 4

Does the integral naturally fit one of the forms $\int \frac{du}{u}$, $\int \frac{du}{a^2+u^2}$, $\int \frac{du}{\sqrt{a^2-u^2}}$, or $\int \frac{du}{u\sqrt{u^2-a^2}}$?

Again, if the ***exact*** variable expression in du is present in the integrand, then you are good to go. If not, move on to option 5.

Examples:

$\int \frac{e^x\,dx}{1+e^{2x}}$ fits the form $\int \frac{du}{a^2+u^2}$.

$\int \frac{e^{2x}\,dx}{81+e^{2x}}$ fits the form $\int \frac{du}{u}$

$\int \frac{\tan(2t)}{\sqrt{\cos^2(2t)-9}}\,dt$ fits the form $\int \frac{du}{u\sqrt{u^2-a^2}}$ (Make sure that you see why!)

Counterexamples:

$$\int \frac{e^{8x}}{1+e^{10x}}\,dx$$

To fit the form $\int \frac{du}{u}$ we'd need $u = 1+e^{10x}$ which would make $du = 10e^{10x}\,dx$. To fit the form $\int \frac{du}{a^2+u^2}$ we'd need $u = e^{5x}$ which would make $du = 5e^{5x}\,dx$. In either case, we have the wrong power of e in the numerator.

$$\int \frac{x^2\,dx}{\sqrt{25-x^2}}$$

This doesn't fit the form $\int \frac{du}{\sqrt{a^2-u^2}}$ because there is no way to account for the factor of x^2 in the numerator. The only other form that might easily fit would be $\int u^{-1/2}\,du$. But this too is a mismatch because if we let $u = 25-x^2$, then $du = -2x\,dx$ and there is an extra factor of x in the numerator. Move on to another option.

$$\int \frac{dt}{e^t\sqrt{e^{2t}-9}}$$

At first glance this looks like it fits $\int \frac{du}{u\sqrt{u^2-a^2}}$ where $u = e^t$. The problem lies with du.

If we let $u = e^t$, then $du = e^t\,dt$. D'oh! There is no factor of e^t in the numerator.

Option 5

Can you introduce variables missing from du? Alternatively, is there too much variable for du that you can resolve by solving your substitution equation for "x"?

Please note that the first strategy probably isn't going to work unless you are trying to fit one of the forms where u is being squared in the denominator, and even then it probably won't work unless "e" is involved.

The second strategy sometimes works in the manner just described. About the only other time it works is if you have a true polynomial being multiplied or divided by a power or radical of a polynomial.

Examples:

$\int \frac{dx}{\sqrt{e^{2x}-9}} = \int \frac{e^x\,dx}{e^x\sqrt{e^{2x}-9}}$ fits the form $\int \frac{du}{u\sqrt{u^2-a^2}}$

$\int (x-5)^2\sqrt{x+2}\,dx = \int (u-7)^2\sqrt{u}\,du$ where $u = x+2$.

(See Option 2 to finish the integration process.)

Option 6

Integration by Parts: $\int u\,dv = u\,v - \int v\,du$.

A Priority List for Choosing the Parts in Integration by Parts

1. A function factor that cannot be antidifferentiated either by itself or in conjunction with other factors **must be** u. Suspect functions include $\ln(x)$, $\sin^{-1}(x)$, $\cos^{-1}(x)$, and $\tan^{-1}(x)$.
2. If there is no non-antidifferentiatable factor but there *is* a polynomial factor then the polynomial factor should be assigned as u.
3. In other situations just split the factors and hope for the best.

Additional Tips on Integration by Parts

- Integration by Parts is a method of "last resort", not a method of "first resort." Before jumping into Integration by Parts make sure that the integral cannot be resolved via substitution.

  Examples: $\int x e^{x^2}\,dx$ is a ***substitution*** integral – you need to consider that option first.

  $\int x e^{x}\,dx$ is an ***I by P*** integral – there are no substitutions to even try.

- If the Integration by Parts method must be repeated while resolving a single integral it is very important that you are consistent in your u and dv assignments. For example, if your integrand contains a "$\sin(x)$" factor that you assign to u and you have to perform Integration by Parts a second time, then you must assign the factor of "$\cos(x)$" to u.