Vector Spaces that emerge from matrices (and affiliated vocabulary)

Suppose that A is and $n \times m$ matrix and that B is the reduced echelon equivalent of A. Then:

- The ***rank*** of A is the number of non-zero rows in B.
- The ***column space*** of A is the set of all linear combinations of the columns of A. The column space of A is a subspace of $\mathbb{R}^n$ and its dimension is equal to $\operatorname{rank}(A)$. The pivot columns of A form a basis for $\operatorname{col}(A)$.
- The ***row space*** of A is the set of all linear combinations of the rows of A. The row space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $\operatorname{rank}(A)$. The non-zero rows of B for a basis for $\operatorname{row}(A)$
- The ***null space*** of A is the set of all solutions to the equation $A\vec{x} = \vec{0}$. The null space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $m - \operatorname{rank}(A)$. One way to find a basis for $\operatorname{nul}(A)$ is to create vectors from the general solution to $A\vec{x} = \vec{0}$ where one vector is created for each free-variable by letting that free variable have a non-zero value whilst all the other free-variables are set to zero.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. State the correct number in each of the blanks below.

The rank of M is ____________________.

The column space of M is a __________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

The row space of M is a _____________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

The null space of M is a _____________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

State a basis for $\text{row}(M)$.

True or false? The stated basis for $\text{row}(M)$ is also a basis for the row space of any matrix that is row equivalent to M. Justify your answer!

State a basis for $\text{col}(M)$.

True or false? The stated basis for $\text{col}(M)$ is also a basis the column space for any matrix that is row equivalent to M. Justify your answer!

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

State a basis for $\operatorname{nul}(M)$.

True or false? The stated basis for $\operatorname{nul}(M)$ is also a basis for the null space of any matrix that is row equivalent to M. Justify your answer!

Example

Find bases for $\operatorname{row}(A)$, $\operatorname{col}(A)$, and $\operatorname{nul}(A)$ where $A=\begin{bmatrix} 2 & -4 & -3 & 17 & 5 \\ -1 & 2 & 3 & -13 & -4 \\ 4 & -8 & 1 & 13 & 3 \end{bmatrix}$.

Example

Consider P_3, the set of all polynomials of degree three or less. Determine whether or not each of the following sets forms a basis for P_3. Justify each answer.

a. $\left\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2\right\}$

b. $\left\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2, t, -1+t^3\right\}$

c. $\left\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2, t\right\}$

Example

Consider P_2, the set of all polynomials of degree two or less. Let V be the set of all polynomials in P_2 that satisfy the equation $\vec{p}(5)=0$. It is easily shown that V forms a subspace of P_2. Answer each of the following questions about V.

a. Find a basis for V and state the dimension of V.

b. Show that $g(x)=3x^2-34x+95\in V$.

c. Express g as a linear combination of the basis stated in part (a).

Example

The set of vectors of form $\begin{bmatrix} 2a & - & 3b & + & c \\ a & + & 3b & + & 5c \\ -3a & + & 2b & - & 4c \end{bmatrix}$ is easily shown to be a subspace of $\mathbb{R}^3$. Determine the dimension of this space.

Suppose that the set $\beta=\{\vec{b}_1,\vec{b}_2,\ldots,\vec{b}_n\}$ forms a basis for $\mathbb{R}^n$. Then for each vector $\vec{x}$ in $\mathbb{R}^n$, there exists a unique set of constants, c_1-c_n such that $\vec{x}=c_1\vec{b}_1+c_2\vec{b}_2+\cdots c_n\vec{b}_n$. The constants c_1-c_n are called the $\beta-$coordinates of $\vec{x}$ and this relationship is symbolized as:

$$[\vec{x}]_\beta=\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}.$$

Example

Consider the $\mathbb{R}^2$ basis $\beta=\left\{\begin{bmatrix}3\\-1\end{bmatrix},\begin{bmatrix}-2\\5\end{bmatrix}\right\}$. Answer each of the following questions relative to this basis.

Determine $\vec{x}$ if $[\vec{x}]_\beta=\begin{bmatrix}-4\\7\end{bmatrix}$.

Determine $[\vec{x}]_\beta$ if $\vec{x}=\begin{bmatrix}26\\-39\end{bmatrix}$.

Theorem

Suppose that β and γ are both bases for $\mathbb{R}^n$ and that $T:\mathbb{R}^n \to \mathbb{R}^n$ by the rule $T\left([\vec{x}]_\beta\right)=[\vec{x}]_\gamma$. Then T is a one-to-one, onto linear transformation and, as such, there exists a matrix $\underset{\gamma \leftarrow \beta}{P}$ with the property that $[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P}[\vec{x}]_\beta$.

Example

Consider the $\mathbb{R}^2$ bases $\beta=\left\{\vec{b}_1,\vec{b}_2\right\}$ and $\gamma=\left\{\vec{c}_1,\vec{c}_2\right\}$ where $\vec{b}_1=3\vec{c}_1+2\vec{c}_2$ and $\vec{b}_2=4\vec{c}_1+3\vec{c}_2$. Answer each of the following questions relative to these bases.

Determine $\underset{\gamma \leftarrow \beta}{P}$.

Determine $[\vec{x}]_\gamma$ if $[\vec{x}]_\beta=\begin{bmatrix}2\\-3\end{bmatrix}$ and verify the results.

Theorem

Suppose that β and γ are two ordered bases for $\mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$, and the components of $\vec{x}$ relative to β are known. Then the components of $\vec{x}$ relative to γ can be determined by the equation $[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$ where $\underset{\gamma \leftarrow \beta}{P}$ is called the ***change-of-coordinates matrix*** from β to γ.

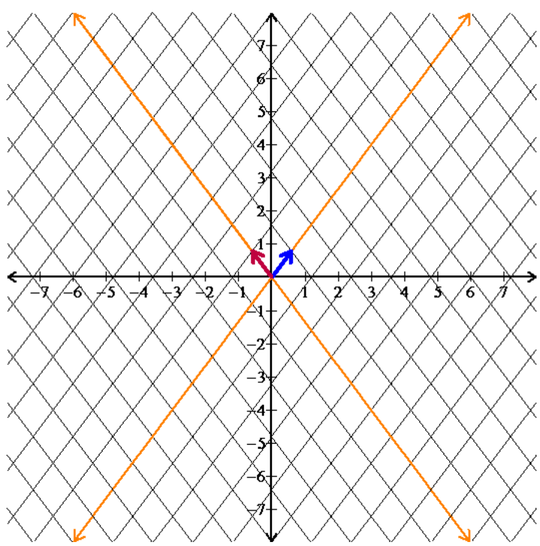
When working in $\mathbb{R}^n$ we can find $\underset{\gamma \leftarrow \beta}{P}$ using Gaussian elimination. Specifically:

$$[\gamma \mid \beta] \xrightarrow{\text{RREF}} \left[I_n \mid \underset{\gamma \leftarrow \beta}{P} \right]$$

Please note that this implies that if β is the standard ordered basis for $\mathbb{R}^n$, then the change-of-basis matrix to γ is simply γ^{-1}.

Example

Let $\vec{c}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and $\gamma = \{ \vec{c}_1, \vec{c}_2 \}$. Find the change=of-basis matrix from the standard basis to γ and use that matrix to find $[\vec{x}]_\gamma$.



Let $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ and $\gamma = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Find the transition matrix from β to γ and use that to find $[\vec{x}]_\gamma$ where $[\vec{x}]_\beta = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$. Verify the result!