Vector Spaces that emerge from matrices (and affiliated vocabulary)

Suppose that A is and $n \times m$ matrix and that B is the reduced echelon equivalent of A. Then:

- The ***rank*** of A is the number of non-zero rows in B.
- The ***column space*** of A is the set of all linear combinations of the columns of A. The column space of A is a subspace of $\mathbb{R}^n$ and its dimension is equal to $\operatorname{rank}(A)$. The pivot columns of A form a basis for $\operatorname{col}(A)$.
- The ***row space*** of A is the set of all linear combinations of the rows of A. The row space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $\operatorname{rank}(A)$. The non-zero rows of B for a basis for $\operatorname{row}(A)$
- The ***null space*** of A is the set of all solutions to the equation $A\vec{x} = \vec{0}$. The null space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $n - \operatorname{rank}(A)$. One way to find a basis for $\operatorname{nul}(A)$ is to create vectors from the general solution to $A\vec{x} = \vec{0}$ where one vector is created for each free-variable by letting that free variable have a non-zero value whilst all the other free-variables are set to zero.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. State the correct number in each of the blanks below.

The rank of M is three.

The column space of M is a 3-dimensional subspace of $\mathbb{R}^{4}$.

The row space of M is a 3-dimensional subspace of $\mathbb{R}^{7}$.

The null space of M is a 4-dimensional subspace of $\mathbb{R}^{7}$.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

State a basis for $\text{row}(M)$.

$\{[1,0,2,0,0,0,5], [0,1,-1,0,0,0,3], [0,0,0,1,0,5,-2]\}$

True or false? The stated basis for $\text{row}(M)$ is also a basis for the row space of any matrix that is row equivalent to M. Justify your answer!

Of course its the same; elementary row operations either change the order of the rows or replace a row with a linear combination of one or two rows. Because row(M) is a vector space, by the closure properties we'll never get out of row(M).

State a basis for $\text{col}(M)$.

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

True or false? The stated basis for $\text{col}(M)$ is also a basis the column space for any matrix that is row equivalent to M. Justify your answer!

I can easilly row-manipulate M into an equivalent matrix where the fourth row is not all zeros ($R1+R4 \to R4$) obviously the last stated basis would not be a basis for the column space of this new matrix

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

State a basis for $\mathbf{nul}(M)$.

The general solution to $M\vec{x} = \vec{0}$ is $\begin{cases} x_1 = -2x_3 - 5x_7 \\ x_2 = x_3 - 3x_7 \\ x_3 \text{ is free} \\ x_4 = -5x_6 + 2x_7 \\ x_5 \text{ is free} \\ x_6 \text{ is free} \\ x_7 \text{ is free} \end{cases}$

A basis for the null space is:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

True or false? The stated basis for $\mathbf{nul}(M)$ is also a basis for the null space of any matrix that is row equivalent to M. Justify your answer!

This is TRUE!

This is the very foundational fact for this entire class! Row manipulating an augmented matrix does not change the solution set to the given system. $[M \mid \vec{0}]$

Example

Find bases for $\mathbf{row}(A)$, $\mathbf{col}(A)$, and $\mathbf{nul}(A)$ where $A=\begin{bmatrix} 2 & -4 & -3 & 17 & 5 \\ -1 & 2 & 3 & -13 & -4 \\ 4 & -8 & 1 & 13 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 2 & -4 & -3 & 17 & 5 \\ -1 & 2 & 3 & -13 & -4 \\ 4 & -8 & 1 & 13 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 4 & 1 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for row(A) is:

$$\left\{ \begin{bmatrix} 1 & -2 & 0 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -3 & -1 \end{bmatrix} \right\}$$

A basis for col(A) is:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix} \right\}$$

The general solution to $A\vec{x}=\vec{0}$ is $\begin{cases} x_1 = 2x_2 - 4x_4 - x_5 \\ x_2 \text{ is free} \\ x_3 = 3x_4 + x_5 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases}$

A basis for nul(A) is:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Example

Consider P_3, the set of all polynomials of degree three or less. Determine whether or not each of the following sets forms a basis for P_3. Justify each answer.

a. $\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2\}$ b. $\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2, t, -1+t^3\}$

c. $\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2, t\}$

An obvious basis for P_3 is $\{1, t, t^2, t^3\}$.
Since the dimension of P_3 is four, sets "a" & "b" are out the door. (Post & Know it).
Set "c" has the proper number of vectors so we only need to determine one of two things, do the vectors span the space or are the vectors linearly independent.
Let's check linear independence.
Since we can only combine like terms when adding polynomials, we can answer the linear independent question by solving,

$$x_1\begin{bmatrix}3\\-6\\0\\1\end{bmatrix}+x_2\begin{bmatrix}0\\3\\7\\-1\end{bmatrix}+x_3\begin{bmatrix}1\\2\\1\\0\end{bmatrix}+x_4\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}3&0&1&0&0\\-6&3&2&1&0\\0&7&1&0&0\\1&-1&0&0&0\end{array}\right]\sim\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{array}\right]$$

Since the trivial solution is the only solution, the four vectors are linearly independent.

Since the dimension of the space is four, the set forms a basis for P_3.

Example

Consider P_2, the set of all polynomials of degree two or less. Let V be the set of all polynomials in P_2 that satisfy the equation $\vec{p}(5)=0$. It is easily shown that V forms a subspace of P_2. Answer each of the following questions about V.

a. Find a basis for V and state the dimension of V.

b. Show that $g(x)=3x^2-34x+95\in V$.

c. Express g as a linear combination of the basis stated in part (a).

a. P_2 is three dimensional. Not every vector in P_2 is in V, Since V is not empty, V must be one-dimensional or two-dimensional.

Two non-zero linearly independent vectors (polynomials) from P_2 that satisfy $\vec{p}(5)=0$ are

$$\vec{p}_1(t) = -5+t \text{ and } \vec{p}_2(t) = 25-10t+t^2$$

Thus, $\{-5+t, 25-10t+t^2\}$ forms a basis for V and V's dimension is two.

b. $g(5) = 75 - 170 + 95$

$= 0$

QED

c. $g(t) = 3t^2 - 34t + 95$

$= 3(25-10t+t^2) + (-4)(-5+t)$

Example

The set of vectors of form $\begin{bmatrix} 2a - 3b + c \\ a + 3b + 5c \\ -3a + 2b - 4c \end{bmatrix}$ is easily shown to be a subspace of $\mathbb{R}^3$. Determine the dimension of this space.

All vectors in this space can be written as:

$$a\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + b\begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix} + c\begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix}.$$

Hence $\left\{ \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix} \right\}$ spans the space. I need to find a basis for the span of that set. An analogous charge, is to find a basis for the column space of $M = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 3 & 5 \\ -3 & 2 & -4 \end{bmatrix}$

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 3 & 5 \\ -3 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \left\{ \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix} \right\}$ form a basis for the give subspace of $\mathbb{R}^3$.

$\therefore$ The dimension of the space is <u>two</u>.

Suppose that the set $\beta = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ forms a basis for $\mathbb{R}^n$. Then for each vector $\vec{x}$ in $\mathbb{R}^n$, there exists a unique set of constants, $c_1 - c_n$ such that $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots c_n\vec{b}_n$. The constants $c_1 - c_n$ are called the β – coordinates of $\vec{x}$ and this relationship is symbolized as:

$$[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Example

Consider the $\mathbb{R}^2$ basis $\beta = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right\}$. Answer each of the following questions relative to this basis.

Determine $\vec{x}$ if $[\vec{x}]_\beta = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$.

$$\vec{x} = -4\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 7\begin{bmatrix} -2 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} -26 \\ 39 \end{bmatrix}$$

Determine $[\vec{x}]_\beta$ if $\vec{x} = \begin{bmatrix} 26 \\ -39 \end{bmatrix}$. (pretend we didn't just see that)

To answer this question we need to solve

$$c_1\begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 26 \\ -39 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & -2 & 26 \\ -1 & 5 & -39 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -7 \end{array}\right]$$

$$\therefore \; [\vec{x}]_\beta = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

Check: $4\begin{bmatrix} 3 \\ -1 \end{bmatrix} + (-7)\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 26 \\ -39 \end{bmatrix}$ ✓

Theorem

Suppose that β and γ are both bases for $\mathbb{R}^n$ and that $T:\mathbb{R}^n \to \mathbb{R}^n$ by the rule $T\left([\vec{x}]_\beta\right) = [\vec{x}]_\gamma$. Then T is a one-to-one, onto linear transformation and, as such, there exists a matrix $\underset{\gamma\leftarrow\beta}{P}$ with the property that $[\vec{x}]_\gamma = \underset{\gamma\leftarrow\beta}{P}[\vec{x}]_\beta$.

Example

Consider the $\mathbb{R}^2$ bases $\beta = \{\vec{b}_1, \vec{b}_2\}$ and $\gamma = \{\vec{c}_1, \vec{c}_2\}$ where $\vec{b}_1 = 3\vec{c}_1 + 2\vec{c}_2$ and $\vec{b}_2 = 4\vec{c}_1 + 3\vec{c}_2$. Answer each of the following questions relative to these bases.

Determine $\underset{\gamma\leftarrow\beta}{P}$.

Define $T(\vec{x}) = [\vec{x}]_\gamma$ and $[\vec{x}]_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{aligned}[\vec{x}]_\gamma &= T(\vec{x}) \\ &= T(x_1\vec{b}_1 + x_2\vec{b}_2) \\ &= T(x_1\vec{b}_1) + T(x_2\vec{b}_2) \\ &= x_1T(\vec{b}_1) + x_2T(\vec{b}_2) \\ &= x_1[\vec{b}_1]_\gamma + x_2[\vec{b}_2]_\gamma \\ &= x_1\begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\end{aligned}$$

$\therefore$ $\underset{\gamma\leftarrow\beta}{P} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$

Determine $[\vec{x}]_\gamma$ if $[\vec{x}]_\beta = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and verify the results.

$$\begin{aligned}[\vec{x}]_\gamma &= \underset{\gamma\leftarrow\beta}{P}[\vec{x}]_\beta \\ &= \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} -6 \\ -5 \end{bmatrix}\end{aligned}$$

Verify

$\vec{x} = 2\vec{b}_1 + (-3)\vec{b}_2$

$= 2(3\vec{c}_1 + 2\vec{c}_2) + (-3)(4\vec{c}_1 + 3\vec{c}_2)$

$= (-6)\vec{c}_1 + (-5)\vec{c}_2$

Theorem

Suppose that β and γ are two ordered bases for $\mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$, and the components of $\vec{x}$ relative to β are known. Then the components of $\vec{x}$ relative to γ can be determined by the equation $[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$ where $\underset{\gamma \leftarrow \beta}{P}$ is called the ***change-of-coordinates matrix*** from β to γ.

When working in $\mathbb{R}^n$ we can find $\underset{\gamma \leftarrow \beta}{P}$ using Gaussian elimination. Specifically:

$$[\gamma \mid \beta] \xrightarrow{\text{RREF}} \left[I_n \;\middle|\; \underset{\gamma \leftarrow \beta}{P} \right]$$

Please note that this implies that if β is the standard ordered basis for $\mathbb{R}^n$, then the change-of-basis matrix to γ is simply γ^{-1}.

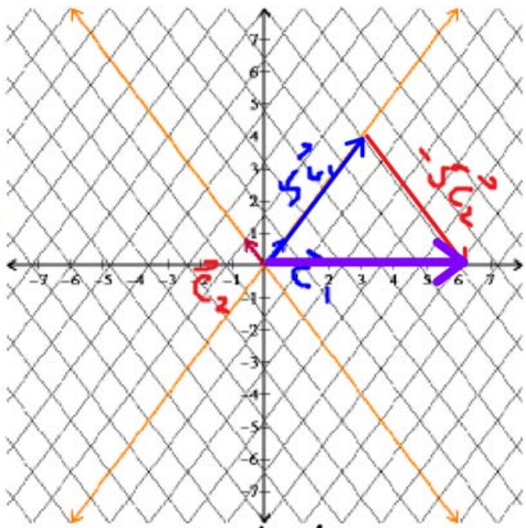
Example

Let $\vec{c}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and $\gamma = \{\vec{c}_1, \vec{c}_2\}$. Find the change=of-basis matrix from the standard basis to γ and use that matrix to find $[\vec{x}]_\gamma$.

$\gamma = \left\{ \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}, \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} \right\}$ $\quad \beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

$$\left[\begin{array}{cc|cc} 3/5 & -3/5 & 1 & 0 \\ 4/5 & 4/5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 5/6 & 5/8 \\ 0 & 1 & -5/6 & 5/8 \end{array}\right]$$

$$\underset{\gamma \leftarrow \beta}{P} = \begin{bmatrix} 5/6 & 5/8 \\ -5/6 & 5/8 \end{bmatrix}$$



Since $\beta = \{\vec{e}_1, \vec{e}_2\}$ and $\vec{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$,

$$[\vec{x}]_\beta = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\therefore \quad [\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta = \begin{bmatrix} 5/6 & 5/8 \\ -5/6 & 5/8 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

& indeed,
$5\vec{c}_1 + (-5)\vec{c}_2 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

Let $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ and $\gamma = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Find the transition matrix from β to γ and use that to find $[\vec{x}]_\gamma$ where $[\vec{x}]_\beta = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$. Verify the result!

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & -2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/3 & 0 & 0 \\ 0 & 1 & 0 & -5/3 & 0 & 1 \\ 0 & 0 & 1 & -2/3 & 1 & 0 \end{array}\right]$$

$$\therefore \underset{\gamma \leftarrow \beta}{P} = \begin{bmatrix} 8/3 & 0 & 0 \\ -5/3 & 0 & 1 \\ -2/3 & 1 & 0 \end{bmatrix}$$

$$[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta = \begin{bmatrix} 8/3 & 0 & 0 \\ -5/3 & 0 & 1 \\ -2/3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ 0 \end{bmatrix}$$

Verification

using the β-coordinates:

$$\vec{x} = 3 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (-4) \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \\ -10 \end{bmatrix}$$

using the γ-coordinates

$$\vec{x} = 8 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (-9) \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \\ -10 \end{bmatrix} \checkmark$$

The coordinates relative to $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are $\begin{bmatrix} 9 \\ -1 \\ -10 \end{bmatrix}$