Vector Spaces that emerge from matrices (and affiliated vocabulary)

Suppose that A is and $n \times m$ matrix and that B is the reduced echelon equivalent of A. Then:

- The ***rank*** of A is the number of non-zero rows in B.
- The ***column space*** of A is the set of all linear combinations of the columns of A. The column space of A is a subspace of $\mathbb{R}^n$ and its dimension is equal to $\operatorname{rank}(A)$. The pivot columns of A form a basis for $\operatorname{col}(A)$.
- The ***row space*** of A is the set of all linear combinations of the rows of A. The row space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $\operatorname{rank}(A)$. The non-zero rows of B for a basis for $\operatorname{row}(A)$
- The ***null space*** of A is the set of all solutions to the equation $A\vec{x} = \vec{0}$. The null space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $m - \operatorname{rank}(A)$. One way to find a basis for $\operatorname{nul}(A)$ is to create vectors from the general solution to $A\vec{x} = \vec{0}$ where one vector is created for each free-variable by letting that free variable have a non-zero value whilst all the other free-variables are set to zero.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. State the correct number in each of the blanks below.

The rank of M is ____________________.

The column space of M is a __________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

The row space of M is a ____________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

The null space of M is a ____________________-dimensional subspace of $\mathbb{R}^{\_\_\_\_}$.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

State a basis for $\operatorname{row}(M)$.

True or false? The stated basis for $\operatorname{row}(M)$ is also a basis for the row space of any matrix that is row equivalent to M. Justify your answer!

State a basis for $\operatorname{col}(M)$.

True or false? The stated basis for $\operatorname{col}(M)$ is also a basis the column space for any matrix that is row equivalent to M. Justify your answer!

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

State a basis for $\operatorname{nul}(M)$.

True or false? The stated basis for $\operatorname{nul}(M)$ is also a basis for the null space of any matrix that is row equivalent to M. Justify your answer!

Example

Find bases for $\operatorname{row}(A)$, $\operatorname{col}(A)$, and $\operatorname{nul}(A)$ where $A = \begin{bmatrix} 2 & -4 & -3 & 17 & 5 \\ -1 & 2 & 3 & -13 & -4 \\ 4 & -8 & 1 & 13 & 3 \end{bmatrix}$.

Example

Consider P_3, the set of all polynomials of degree three or less. Determine whether or not each of the following sets forms a basis for P_3. Justify each answer.

a. $\left\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2\right\}$

b. $\left\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2, t, -1+t^3\right\}$

c. $\left\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2, t\right\}$

Example

Consider P_2, the set of all polynomials of degree two or less. Let V be the set of all polynomials in P_2 that satisfy the equation $\vec{p}(5)=0$. It is easily shown that V forms a subspace of P_2. Answer each of the following questions about V.

a. Find a basis for V and state the dimension of V.

b. Show that $g(x)=3x^2-34x+95\in V$.

c. Express g as a linear combination of the basis stated in part (a).

Example

The set of vectors of form $\begin{bmatrix} 2a - 3b + c \\ a + 3b + 5c \\ -3a + 2b - 4c \end{bmatrix}$ is easily shown to be a subspace of $\mathbb{R}^3$. Determine the dimension of this space.