Vector Spaces that emerge from matrices (and affiliated vocabulary)

Suppose that A is and $n \times m$ matrix and that B is the reduced echelon equivalent of A. Then:

- The ***rank*** of A is the number of non-zero rows in B.
- The ***column space*** of A is the set of all linear combinations of the columns of A. The column space of A is a subspace of $\mathbb{R}^n$ and its dimension is equal to $\operatorname{rank}(A)$. The pivot columns of A form a basis for $\operatorname{col}(A)$.
- The ***row space*** of A is the set of all linear combinations of the rows of A. The row space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $\operatorname{rank}(A)$. The non-zero rows of B for a basis for $\operatorname{row}(A)$
- The ***null space*** of A is the set of all solutions to the equation $A\vec{x} = \vec{0}$. The null space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $m - \operatorname{rank}(A)$. One way to find a basis for $\operatorname{nul}(A)$ is to create vectors from the general solution to $A\vec{x} = \vec{0}$ where one vector is created for each free-variable by letting that free variable have a non-zero value whilst all the other free-variables are set to zero.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. State the correct number in each of the blanks below.

The rank of M is ____3____.

The column space of M is a ____3____-dimensional subspace of $\mathbb{R}^{4}$.

The row space of M is a ____3____-dimensional subspace of $\mathbb{R}^{7}$.

The null space of M is a ____4____-dimensional subspace of $\mathbb{R}^{7}$.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

rref

State a basis for $\text{row}(M)$.

$\{[1,0,2,0,0,0,5], [0,1,-1,0,0,0,3], [0,0,0,1,0,5,-2]\}$

True or false? The stated basis for $\text{row}(M)$ is also a basis for the row space of any matrix that is row equivalent to M. Justify your answer!

Row equivalent matrices are created via elementary row operations which are either linear combinations of rows or swapping rows.
So, if M is row equivalent to N,
$\text{row}(M) = \text{row}(N)$

State a basis for $\text{col}(M)$.

Since M was already in RREF,
$\text{Basis}(\text{col}(M)) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

True or false? The stated basis for $\text{col}(M)$ is also a basis the column space for any matrix that is row equivalent to M. Justify your answer!

This is <u>so</u> false!

$M \quad R_1 + R_4 \to R_4 \quad \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 1 & 0 & 2 & 0 & 0 & 0 & 5 \end{bmatrix}$

and obviously C_1 is not in the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

State a basis for $\mathbf{nul}(M)$.

$$\begin{cases} x_1 = -2x_3 - 5x_7 \\ x_2 = x_3 - 3x_7 \\ x_4 = -5x_6 + 2x_7 \\ x_3, x_5, x_6, x_7 \text{ are free} \end{cases}$$

Basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

True or false? The stated basis for $\mathbf{nul}(M)$ is also a basis for the null space of any matrix that is row equivalent to M. Justify your answer!

This is true, it's the entire basis for this class. Elementary row operations do not affect the solution set to a system of equations.

$M\vec{x} = \vec{0}$

Example

Find bases for $\mathbf{row}(A)$, $\mathbf{col}(A)$, and $\mathbf{nul}(A)$ where $A = \begin{bmatrix} 2 & -4 & -3 & 17 & 5 \\ -1 & 2 & 3 & -13 & -4 \\ 4 & -8 & 1 & 13 & 3 \end{bmatrix}$.

$$A \sim \begin{bmatrix} 1 & -2 & 0 & 4 & 1 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Basis}(\text{row}(A)) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ -1 \end{bmatrix} \right\}$$

$$\text{Basis}(\text{col}(A)) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix} \right\}$$

$$A\vec{x} = \vec{0}$$

$$\begin{cases} x_1 = 2x_2 - 4x_4 - x_5 \\ x_3 = 3x_4 + x_5 \\ x_2, x_4, x_5 \text{ are free} \end{cases}$$

$$\text{Basis}(\text{nul}(A)) = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Example

Consider P_3, the set of all polynomials of degree three or less. Determine whether or not each of the following sets forms a basis for P_3. Justify each answer.

a. $\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2\}$ b. $\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2, t, -1+t^3\}$

c. $\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2, t\}$

A basis for P_3 is $\{t^3, t^2, t, 1\}$, so $\dim(P_3)=4$ which tosses "a" and "b".

For c, lets define $T: P_3 \to \mathbb{R}^4 \ni$

$$T(a+bt+ct^2+dt^3) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Let $A = \begin{bmatrix} 3 & 0 & 1 & 0 \\ -6 & 3 & 2 & 1 \\ 0 & 7 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$

If $\text{col}(A) = \mathbb{R}^4$, "set c" is a basis for P_3; if not, not.

$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, ergo "set c" is a basis for P_3

Let's show that $1+t+t^2+t^3 \in \text{span}(\text{"set c"})$

$$\left[\begin{array}{cccc|c} 3 & 0 & 1 & 0 & 1 \\ -6 & 3 & 2 & 1 & 1 \\ 0 & 7 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7/4 \\ 0 & 1 & 0 & 0 & 3/4 \\ 0 & 0 & 1 & 0 & -17/4 \\ 0 & 0 & 0 & 1 & 71/4 \end{array}\right]$$

check $\frac{7}{4}(3-6t+t^3) + \frac{3}{4}(3t+7t^2-t^3) - \frac{17}{4}(1+2t+t^2) + \frac{71}{4}(t)$
$= 1+t+t^2+t^3$ ✓

Example

Consider P_2, the set of all polynomials of degree two or less. Let V be the set of all polynomials in P_2 that satisfy the equation $\vec{p}(5)=0$. It is easily shown that V forms a subspace of P_2. Answer each of the following questions about V.

a. Find a basis for V and state the dimension of V.

b. Show that $g(x)=3x^2-34x+95 \in V$.

c. Express g as a linear combination of the basis stated in part (a).

The dimension of P_2 is 3 (Basis: $\{t^2, t, 1\}$)

V is not all of P_2 ($f(x)=x+7 \notin V$)

Ergo, the dimension of V is one or two.

$b_1(x) = x-5 \in V$

$b_2(x) = x^2-10x+25 \in V$

$b_1(x)$ and $b_2(x)$ are obviously linearly independent

$c_1(x^2-10x+25) + c_2(x-5) = 0$

$c_2 = 0$

$c_1 = 0$

$\therefore$ $\{b_1(x), b_2(x)\}$ is a basis for V.

b. $g(5) = 3(25) - 34(5) + 95 = 0$ QED

c. $\left[\begin{array}{cc|c} -5 & 25 & 95 \\ 1 & -10 & -34 \\ 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$

$g(x) = -4\,b_1(x) + 3\,b_2(x)$

For Real

$-4(x-5) + 3(x^2-10x+25) = 3x^2-34x+95$ ✓

Example

The set of vectors of form $\begin{bmatrix} 2a & - & 3b & + & c \\ a & + & 3b & + & 5c \\ -3a & + & 2b & - & 4c \end{bmatrix}$ is easily shown to be a subspace of $\mathbb{R}^3$. Determine the dimension of this space.

A spanning set for the space is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix} \right\},$$ so a basis for the set is a basis for col(B)

where $B = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 3 & 5 \\ -3 & 2 & -4 \end{bmatrix}$

$$B \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ a basis for the given set is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix} \right\}$$

$\therefore$ The dimension of the space is <u>two</u>.