Suppose that the set $\beta = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ forms a basis for $\mathbb{R}^n$. Then for each vector $\vec{x}$ in $\mathbb{R}^n$, there exists a unique set of constants, $c_1 - c_n$ such that $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots c_n\vec{b}_n$. The constants $c_1 - c_n$ are called the $\beta-$coordinates of $\vec{x}$ and this relationship is symbolized as:

$$[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Example

Consider the $\mathbb{R}^2$ basis $\beta = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right\}$. Answer each of the following questions relative to this basis.

Determine $\vec{x}$ if $[\vec{x}]_\beta = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$.

Determine $[\vec{y}]_\beta$ if $\vec{y} = \begin{bmatrix} 26 \\ -39 \end{bmatrix}$.

Theorem

Suppose that β and γ are both bases for $\mathbb{R}^n$ and that $T:\mathbb{R}^n \to \mathbb{R}^n$ by the rule $T\left(\left[\vec{x}\right]_\beta\right) = \left[\vec{x}\right]_\gamma$. Then T is a one-to-one, onto linear transformation and, as such, there exists a matrix $\underset{\gamma \leftarrow \beta}{P}$ with the property that $\left[\vec{x}\right]_\gamma = \underset{\gamma \leftarrow \beta}{P}\left[\vec{x}\right]_\beta$.

Example

Consider the $\mathbb{R}^2$ bases $\beta = \left\{\vec{b}_1, \vec{b}_2\right\}$ and $\gamma = \left\{\vec{c}_1, \vec{c}_2\right\}$ where $\vec{b}_1 = 3\vec{c}_1 + 2\vec{c}_2$ and $\vec{b}_2 = 4\vec{c}_1 + 3\vec{c}_2$. Answer each of the following questions relative to these bases.

Determine $\underset{\gamma \leftarrow \beta}{P}$.

Determine $\left[\vec{x}\right]_\gamma$ if $\left[\vec{x}\right]_\beta = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and verify the results.

Theorem

Suppose that β and γ are two ordered bases for $\mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$, and the components of $\vec{x}$ relative to β are known. Then the components of $\vec{x}$ relative to γ can be determined by the equation $[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$ where $\underset{\gamma \leftarrow \beta}{P}$ is called the ***change-of-coordinates matrix*** from β to γ.

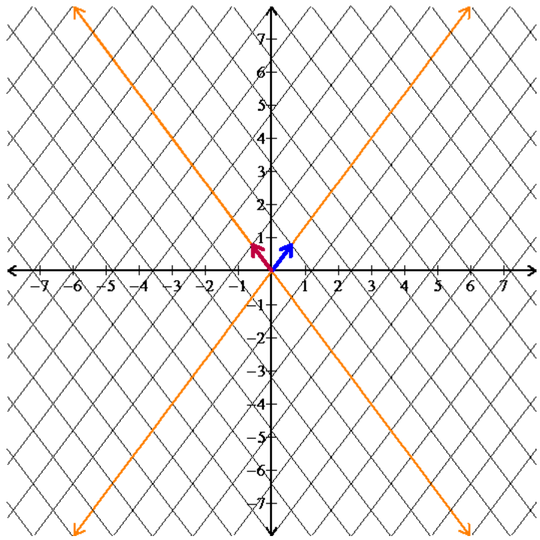
When working in $\mathbb{R}^n$ we can find $\underset{\gamma \leftarrow \beta}{P}$ using Gaussian elimination. Specifically:

$$[\gamma \mid \beta] \xrightarrow{\text{RREF}} \left[I_n \mid \underset{\gamma \leftarrow \beta}{P} \right]$$

Please note that this implies that if β is the standard ordered basis for $\mathbb{R}^n$, then the change-of-basis matrix to γ is simply γ^{-1}.

Example

Let $\vec{c}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and $\gamma = \{ \vec{c}_1, \vec{c}_2 \}$. Find the change=of-basis matrix from the standard basis to γ and use that matrix to find $[\vec{x}]_\gamma$.



Let $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ and $\gamma = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Find the transition matrix from β to γ and use that to find $[\vec{x}]_\gamma$ where $[\vec{x}]_\beta = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$. Verify the result!