Suppose that the set $\beta=\{\vec{b}_1,\vec{b}_2,\ldots,\vec{b}_n\}$ forms a basis for $\mathbb{R}^n$. Then for each vector $\vec{x}$ in $\mathbb{R}^n$, there exists a unique set of constants, $c_1 - c_n$ such that $\vec{x}=c_1\vec{b}_1+c_2\vec{b}_2+\cdots c_n\vec{b}_n$. The constants c_1-c_n are called the $\beta-$coordinates of $\vec{x}$ and this relationship is symbolized as:

$$[\vec{x}]_\beta=\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Example

Consider the $\mathbb{R}^2$ basis $\beta=\left\{\begin{bmatrix}3\\-1\end{bmatrix},\begin{bmatrix}-2\\5\end{bmatrix}\right\}$. Answer each of the following questions relative to this basis.

Determine $\vec{x}$ if $[\vec{x}]_\beta=\begin{bmatrix}-4\\7\end{bmatrix}$.

"$\vec{x}$" are the $\{\vec{e}_1,\vec{e}_2\}$ coordinates, i.e. what $\vec{x}$ has always been.

$\begin{bmatrix}-4\\7\end{bmatrix}$ are the β coordinates, i.e.

$$\vec{x} = -4\begin{bmatrix}3\\-1\end{bmatrix}+7\begin{bmatrix}-2\\5\end{bmatrix}$$
$$=\begin{bmatrix}-26\\39\end{bmatrix}$$

Determine $[\vec{y}]_\beta$ if $\vec{y}=\begin{bmatrix}26\\-39\end{bmatrix}$. By observation, given the last example, $[\vec{y}]_\beta=\begin{bmatrix}4\\-7\end{bmatrix}$

If we couldn't make that observation, however, we need to find scalars, c_1 & c_2, such that

$$c_1\begin{bmatrix}3\\-1\end{bmatrix}+c_2\begin{bmatrix}-2\\5\end{bmatrix}=\begin{bmatrix}26\\-39\end{bmatrix}$$

$$\left[\begin{array}{cc|c}3 & -2 & 26\\ -1 & 5 & -39\end{array}\right]\sim\left[\begin{array}{cc|c}1 & 0 & 4\\ 0 & 1 & -7\end{array}\right] \therefore [\vec{y}]_\beta=\begin{bmatrix}4\\-7\end{bmatrix}$$ ✓

Theorem

Suppose that β and γ are both bases for $\mathbb{R}^n$ and that $T:\mathbb{R}^n \to \mathbb{R}^n$ by the rule $T\left(\left[\vec{x}\right]_\beta\right)=\left[\vec{x}\right]_\gamma$. Then T is a one-to-one, onto linear transformation and, as such, there exists a matrix $\underset{\gamma \leftarrow \beta}{P}$ with the property that $\left[\vec{x}\right]_\gamma = \underset{\gamma \leftarrow \beta}{P}\left[\vec{x}\right]_\beta$.

Example

Consider the $\mathbb{R}^2$ bases $\beta=\left\{\vec{b}_1,\vec{b}_2\right\}$ and $\gamma=\left\{\vec{c}_1,\vec{c}_2\right\}$ where $\vec{b}_1=3\vec{c}_1+2\vec{c}_2$ and $\vec{b}_2=4\vec{c}_1+3\vec{c}_2$. Answer each of the following questions relative to these bases.

Determine $\underset{\gamma \leftarrow \beta}{P}$.

Determine $\left[\vec{x}\right]_\gamma$ if $\left[\vec{x}\right]_\beta=\begin{bmatrix}2\\-3\end{bmatrix}$ and verify the results.

Theorem

Suppose that β and γ are two ordered bases for $\mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$, and the components of $\vec{x}$ relative to β are known. Then the components of $\vec{x}$ relative to γ can be determined by the equation $[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P}[\vec{x}]_\beta$ where $\underset{\gamma \leftarrow \beta}{P}$ is called the ***change-of-coordinates matrix*** from β to γ.

When working in $\mathbb{R}^n$ we can find $\underset{\gamma \leftarrow \beta}{P}$ using Gaussian elimination. Specifically:

Basis whose coordinates are unknown → γ ; Basis whose coordinates are known → β

$$[\gamma \mid \beta] \xrightarrow{\text{RREF}} \left[I_n \mid \underset{\gamma \leftarrow \beta}{P} \right]$$

Please note that this implies that if β is the standard ordered basis for $\mathbb{R}^n$, then the change-of-basis matrix to γ is simply γ^{-1}.

Example

Let $\vec{c}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and $\gamma = \{\vec{c}_1, \vec{c}_2\}$. Find the change=of-basis matrix from the standard basis to γ and use that matrix to find $[\vec{x}]_\gamma$.

Define $\sigma = \{\vec{e}_1, \vec{e}_2\}$. Then $\vec{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \Rightarrow [\vec{x}]_\sigma = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

i.e. $\vec{x} = 6\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$.

new basis ↘ ; ↙ old basis

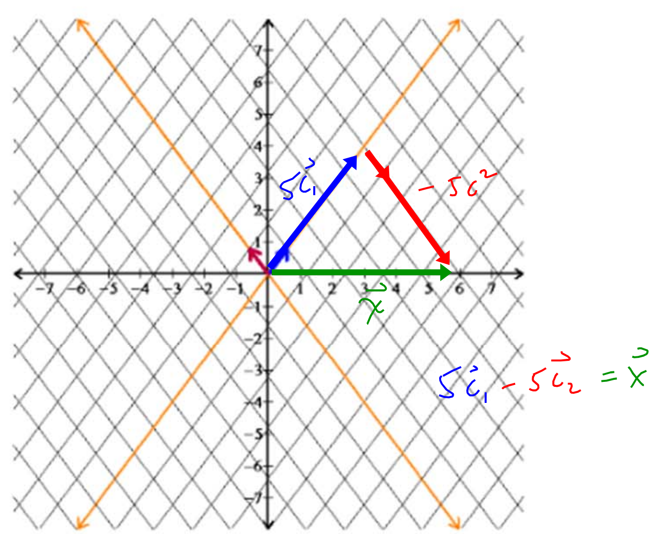
$$\left[\begin{array}{cc|cc} 3/5 & -3/5 & 1 & 0 \\ 4/5 & 4/5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 5/6 & 5/8 \\ 0 & 1 & -5/6 & 5/8 \end{array}\right]$$

$$\therefore \underset{\gamma \leftarrow \sigma}{P} = \begin{bmatrix} 5/6 & 5/8 \\ -5/6 & 5/8 \end{bmatrix}$$

Given: $[\vec{x}]_\sigma = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$$[\vec{x}]_\gamma = \underset{\gamma \leftarrow \sigma}{P}[\vec{x}]_\sigma = \begin{bmatrix} 5/6 & 5/8 \\ -5/6 & 5/8 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

i.e. $\vec{x} = 5\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} + (-5)\begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$



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$$[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$$

Let $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ and $\gamma = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Find the transition matrix from β to γ and use that to find $[\vec{x}]_\gamma$ where $[\vec{x}]_\beta = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$. Verify the result!

unknown — known

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & -2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/3 & 0 & 0 \\ 0 & 1 & 0 & -5/3 & 0 & 1 \\ 0 & 0 & 1 & -2/3 & 1 & 0 \end{array}\right]$$

$$\therefore \underset{\gamma \leftarrow \beta}{P} = \begin{bmatrix} 8/3 & 0 & 0 \\ -5/3 & 0 & 1 \\ -2/3 & 1 & 0 \end{bmatrix}$$

$$\therefore [\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta = \begin{bmatrix} 8/3 & 0 & 0 \\ -5/3 & 0 & 1 \\ -2/3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ 0 \end{bmatrix}$$

Check:

β-coordinates: $\vec{x} = (3)\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + (2)\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (-4)\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \\ -10 \end{bmatrix}$

γ-coordinates: $\vec{x} = (8)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (-9)\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + (0)\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \\ -10 \end{bmatrix}$ ✓

$$(9)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-10)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$9\hat{i} + (-1)\hat{j} + (-10)\hat{k}$$