Suppose that the set $\beta=\{\vec{b}_1,\vec{b}_2,\ldots,\vec{b}_n\}$ forms a basis for $\mathbb{R}^n$. Then for each vector $\vec{x}$ in $\mathbb{R}^n$, there exists a unique set of constants, c_1-c_n such that $\vec{x}=c_1\vec{b}_1+c_2\vec{b}_2+\cdots c_n\vec{b}_n$. The constants c_1-c_n are called the $\beta-$coordinates of $\vec{x}$ and this relationship is symbolized as:

$$[\vec{x}]_\beta=\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}.$$

Example

Consider the $\mathbb{R}^2$ basis $\beta=\left\{\begin{bmatrix}3\\-1\end{bmatrix},\begin{bmatrix}-2\\5\end{bmatrix}\right\}$. Answer each of the following questions relative to this basis.

Determine $\vec{x}$ if $[\vec{x}]_\beta=\begin{bmatrix}-4\\7\end{bmatrix}$.

" $\vec{x}$ " is the $\{\vec{e}_1,\vec{e}_2\}$-coordinates (i.e. standard coordinates)

$$[\vec{x}]_\beta=\begin{bmatrix}-4\\7\end{bmatrix} \Rightarrow \vec{x}=(-4)\begin{bmatrix}3\\-1\end{bmatrix}+7\begin{bmatrix}-2\\5\end{bmatrix}=\begin{bmatrix}-26\\39\end{bmatrix}$$

Determine $[\vec{x}]_\beta$ if $\vec{x}=\begin{bmatrix}26\\-39\end{bmatrix}$.

We want constants, x_1 & x_2, such that

$$x_1\begin{bmatrix}3\\-1\end{bmatrix}+x_2\begin{bmatrix}-2\\5\end{bmatrix}=\begin{bmatrix}26\\-39\end{bmatrix}$$

$$\left[\begin{array}{cc|c}3&-2&26\\-1&5&-39\end{array}\right]\sim\left[\begin{array}{cc|c}1&0&4\\0&1&-7\end{array}\right]$$

$$\vec{x}=\begin{bmatrix}26\\-39\end{bmatrix}\Rightarrow[\vec{x}]_\beta=\begin{bmatrix}4\\-7\end{bmatrix}$$

Theorem

Suppose that β and γ are both bases for $\mathbb{R}^n$ and that $T:\mathbb{R}^n \to \mathbb{R}^n$ by the rule $T\left([\vec{x}]_\beta\right) = [\vec{x}]_\gamma$. Then T is a one-to-one, onto linear transformation and, as such, there exists a matrix $\underset{\gamma\leftarrow\beta}{P}$ with the property that $[\vec{x}]_\gamma = \underset{\gamma\leftarrow\beta}{P}[\vec{x}]_\beta$.

Example

Consider the $\mathbb{R}^2$ bases $\beta = \{\vec{b}_1, \vec{b}_2\}$ and $\gamma = \{\vec{c}_1, \vec{c}_2\}$ where $\vec{b}_1 = 3\vec{c}_1 + 2\vec{c}_2$ and $\vec{b}_2 = 4\vec{c}_1 + 3\vec{c}_2$. Answer each of the following questions relative to these bases.

Determine $\underset{\gamma\leftarrow\beta}{P}$. If $[\vec{v}]_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then we know that $\vec{v} = x_1\vec{b}_1 + x_2\vec{b}_2$. We need to find $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ such that $[\vec{v}]_\gamma = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$; ie. $\vec{v} = y_1\vec{c}_1 + y_2\vec{c}_2$.

$$\vec{v} = x_1\vec{b}_1 + x_2\vec{b}_2$$
$$= x_1(3\vec{c}_1 + 2\vec{c}_2) + x_2(4\vec{c}_1 + 3\vec{c}_2)$$
$$= \underbrace{(3x_1 + 4x_2)\vec{c}_1 + (2x_1 + 3x_2)\vec{c}_2}_{\gamma\text{-coordinates of } \vec{v}}$$

$$[\vec{v}]_\gamma = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}[\vec{v}]_\beta$$

$$\underset{\gamma\leftarrow\beta}{P} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

Determine $[\vec{x}]_\gamma$ if $[\vec{x}]_\beta = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and verify the results.

$$[\vec{x}]_\gamma = \underset{\gamma\leftarrow\beta}{P}[\vec{x}]_\beta = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \end{bmatrix}$$

$$[\vec{x}]_\beta = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\vec{x} = 2\vec{b}_1 + (-3)\vec{b}_2$$
$$= 2(3\vec{c}_1 + 2\vec{c}_2) + (-3)(4\vec{c}_1 + 3\vec{c}_2)$$
$$= -6\vec{c}_1 + (-5)\vec{c}_2$$

$$[\vec{x}]_\gamma = \begin{bmatrix} -6 \\ -5 \end{bmatrix}$$

$$\vec{x} = -6\vec{c}_1 + (-5)\vec{c}_2$$

Theorem

Suppose that β and γ are two ordered bases for $\mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$, and the components of $\vec{x}$ relative to β are known. Then the components of $\vec{x}$ relative to γ can be determined by the equation $[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$ where $\underset{\gamma \leftarrow \beta}{P}$ is called the ***change-of-coordinates matrix*** from β to γ.

When working in $\mathbb{R}^n$ we can find $\underset{\gamma \leftarrow \beta}{P}$ using Gaussian elimination. Specifically:

new   Known

$$[\gamma \mid \beta] \xrightarrow{\text{RREF}} \left[I_n \;\middle|\; \underset{\gamma \leftarrow \beta}{P} \right]$$

Please note that this implies that if β is the standard ordered basis for $\mathbb{R}^n$, then the change-of-basis matrix to γ is simply γ^{-1}.

Example

Let $\vec{c}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and $\gamma = \{\vec{c}_1, \vec{c}_2\}$. Find the change=of-basis matrix from the standard basis to γ and use that matrix to find $[\vec{x}]_\gamma$.

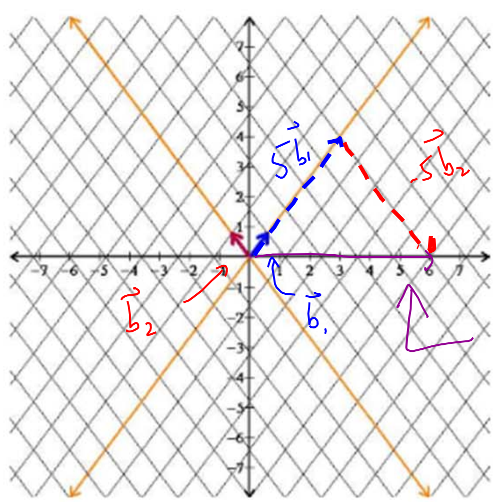
new basis   from basis

$$\left[\begin{array}{cc|cc} 3/5 & -3/5 & 1 & 0 \\ 4/5 & 4/5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 5/6 & 5/8 \\ 0 & 1 & -5/6 & 5/8 \end{array}\right]$$

$$\therefore \underbrace{P_\beta}_{\beta \leftarrow \{\hat{e}_1, \hat{e}_2\}} = \begin{bmatrix} 5/6 & 5/8 \\ -5/6 & 5/8 \end{bmatrix}$$

For $\vec{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$$[\vec{x}]_\beta = P_\beta \vec{x} = \begin{bmatrix} 5/6 & 5/8 \\ -5/6 & 5/8 \end{bmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$



$5\vec{b} + (-5\vec{b}_2) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

woohoo!

Let $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ and $\gamma = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Find the transition matrix from β to γ and use that to find $[\vec{x}]_\gamma$ where $[\vec{x}]_\beta = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$. Verify the result!

To From

$$[\gamma \,\vdots\, \beta] = \left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & -2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/3 & 0 & 0 \\ 0 & 1 & 0 & -5/3 & 0 & 1 \\ 0 & 0 & 1 & -2/3 & 1 & 0 \end{array}\right]$$

$$\therefore \quad {}_{\gamma \leftarrow \beta}P = \frac{1}{3}\begin{bmatrix} 8 & 0 & 0 \\ -5 & 0 & 3 \\ -2 & 3 & 0 \end{bmatrix}$$

$$[\vec{x}]_\beta = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \quad \Rightarrow [\vec{x}]_\gamma = {}_{\gamma \leftarrow \beta}P \, [\vec{x}]_\beta$$

$$= \frac{1}{3}\begin{bmatrix} 8 & 0 & 0 \\ -5 & 0 & 3 \\ -2 & 3 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ -9 \\ 0 \end{bmatrix}$$

$$[\vec{x}]_\beta = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

$$\vec{x} = 3\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (-4)\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ -1 \\ -10 \end{bmatrix}$$

$$[\vec{x}]_\gamma = \begin{bmatrix} 8 \\ -9 \\ 0 \end{bmatrix}$$

$$\vec{x} = 8\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (-9)\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ -1 \\ -10 \end{bmatrix} \quad \checkmark\checkmark\checkmark$$