

Suppose that the set $\beta = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ forms a basis for \mathbb{R}^n . Then for each vector \vec{x} in \mathbb{R}^n , there exists a unique set of constants, $c_1 - c_n$ such that $\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \dots + c_n \vec{b}_n$. The constants $c_1 - c_n$ are called the β -coordinates of \vec{x} and this relationship is symbolized as:

$$[\vec{x}]_{\beta} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Example

Consider the \mathbb{R}^2 basis $\beta = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right\}$. Answer each of the following questions relative to this basis.

Determine \vec{x} if $[\vec{x}]_{\beta} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$.

by "x" we mean $[\vec{x}]_{\{\vec{e}_1, \vec{e}_2\}}$

$$\vec{x} = -4 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + 7 \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -26 \\ 39 \end{bmatrix}$$

Determine $[\vec{x}]_{\beta}$ if $\vec{x} = \begin{bmatrix} 26 \\ -39 \end{bmatrix}$. translation: solve $c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 26 \\ -39 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 3 & -2 & 26 \\ -1 & 5 & -39 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -7 \end{array} \right]$$

$$\therefore [\vec{x}]_{\beta} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

$$\text{check: } 4 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + (-7) \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 26 \\ -39 \end{bmatrix} \quad \checkmark$$

$$\begin{aligned}
 & 2\vec{b}_1 - 3\vec{b}_2 \\
 &= 2(3\vec{c}_1 + 2\vec{c}_2) - 3(4\vec{c}_1 + 3\vec{c}_2) \\
 &= -6\vec{c}_1 - 5\vec{c}_2 \quad \checkmark
 \end{aligned}$$

Theorem

Suppose that β and γ are both bases for \mathbb{R}^n and that $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by the rule $T([\vec{x}]_\beta) = [\vec{x}]_\gamma$. Then T is a one-to-one, onto linear transformation and, as such, there exists a matrix $P_{\gamma \leftarrow \beta}$ with the property that $[\vec{x}]_\gamma = P_{\gamma \leftarrow \beta} [\vec{x}]_\beta$.

Example

Consider the \mathbb{R}^2 bases $\beta = \{\vec{b}_1, \vec{b}_2\}$ and $\gamma = \{\vec{c}_1, \vec{c}_2\}$ where $\vec{b}_1 = 3\vec{c}_1 + 2\vec{c}_2$ and $\vec{b}_2 = 4\vec{c}_1 + 3\vec{c}_2$. Answer each of the following questions relative to these bases.

Determine $P_{\gamma \leftarrow \beta}$. Known coordinates: β , unknown coordinates: γ
 we know $[\vec{x}]_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we want to know $[\vec{x}]_\gamma = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
 how... $\vec{x} = x_1 \vec{b}_1 + x_2 \vec{b}_2$
 $= x_1(3\vec{c}_1 + 2\vec{c}_2) + x_2(4\vec{c}_1 + 3\vec{c}_2)$
 $= (3x_1 + 4x_2)\vec{c}_1 + (2x_1 + 3x_2)\vec{c}_2$
 $\therefore [\vec{x}]_\gamma = \begin{bmatrix} 3x_1 + 4x_2 \\ 2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
 $\therefore P_{\gamma \leftarrow \beta} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$

Determine $[\vec{x}]_\gamma$ if $[\vec{x}]_\beta = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and verify the results.

$$[\vec{x}]_\gamma = P_{\gamma \leftarrow \beta} [\vec{x}]_\beta = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \end{bmatrix}$$

Check
 $\vec{x} = 2\vec{b}_1 + (-3)\vec{b}_2$ using $[\vec{x}]_\beta = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

using $[\vec{x}]_\gamma = \begin{bmatrix} -6 \\ -5 \end{bmatrix}$, $\vec{x} = (-6)\vec{c}_1 + (-5)\vec{c}_2$
 what we need to check, that $2\vec{b}_1 - 3\vec{b}_2 = -6\vec{c}_1 - 5\vec{c}_2$

Theorem

Suppose that β and γ are two ordered bases for \mathbb{R}^n , $\vec{x} \in \mathbb{R}^n$, and the components of \vec{x} relative to β are known. Then the components of \vec{x} relative to γ can be determined by the equation

$$[\vec{x}]_{\gamma} = P_{\gamma \leftarrow \beta} [\vec{x}]_{\beta} \text{ where } P_{\gamma \leftarrow \beta} \text{ is called the } \underline{\text{change-of-coordinates matrix}} \text{ from } \beta \text{ to } \gamma.$$

When working in \mathbb{R}^n we can find $P_{\gamma \leftarrow \beta}$ using Gaussian elimination. Specifically:

$$[\gamma \mid \beta] \xrightarrow{\text{RREF}} \left[I_n \mid P_{\gamma \leftarrow \beta} \right]$$

Please note that this implies that if β is the standard ordered basis for \mathbb{R}^n , then the change-of-basis matrix to γ is simply γ^{-1} .

Example

Let $\vec{c}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and $\gamma = \{\vec{c}_1, \vec{c}_2\}$. Find the change-of-basis matrix from the standard basis to γ and use that matrix to find $[\vec{x}]_{\gamma}$.

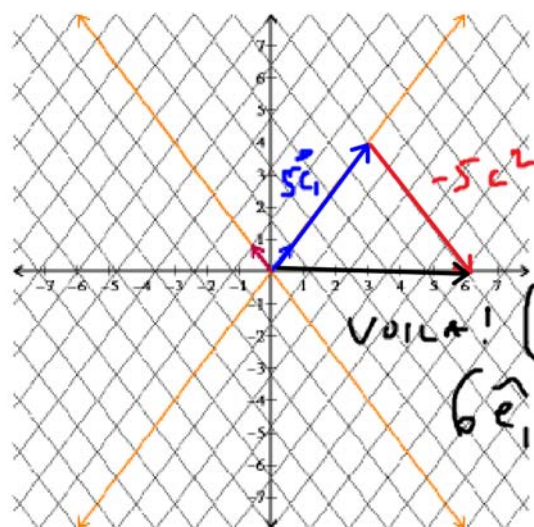
new basis: $\left\{ \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}, \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} \right\}$: original basis: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$$\begin{array}{c} \text{NEW} \quad \text{ORIGINAL} \\ \left[\begin{array}{cc|cc} 3/5 & -3/5 & 1 & 0 \\ 4/5 & 4/5 & 0 & 1 \end{array} \right] \\ \sim \left[\begin{array}{cc|cc} 1 & 0 & 5/6 & 5/8 \\ 0 & 1 & -5/6 & 5/8 \end{array} \right] \end{array}$$

$$P_{\gamma \leftarrow \{\vec{e}_1, \vec{e}_2\}} = \begin{bmatrix} 5/6 & 5/8 \\ -5/6 & 5/8 \end{bmatrix}$$

$$[\vec{x}]_{\gamma} = P_{\gamma \leftarrow \{\vec{e}_1, \vec{e}_2\}} \vec{x}$$

$$\begin{aligned} &= \begin{bmatrix} 5/6 & 5/8 \\ -5/6 & 5/8 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ -5 \end{bmatrix} \end{aligned}$$



Let $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ and $\gamma = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Find the transition matrix from β to γ and

use that to find $[\vec{x}]_\gamma$ where $[\vec{x}]_\beta = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$. Verify the result!

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & -2 & 2 & 2 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/3 & 0 & 0 \\ 0 & 1 & 0 & -5/3 & 0 & 1 \\ 0 & 0 & 1 & -2/3 & 1 & 0 \end{array} \right]$$

$$\therefore P_{\beta \leftarrow \gamma} = \begin{bmatrix} 8/3 & 0 & 0 \\ -5/3 & 0 & 1 \\ -2/3 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} [\vec{x}]_\gamma &= P_{\beta \leftarrow \gamma} [\vec{x}]_\beta \\ &= \begin{bmatrix} 8/3 & 0 & 0 \\ -5/3 & 0 & 1 \\ -2/3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ -9 \\ 0 \end{bmatrix} \end{aligned}$$

Check

$$\beta: \vec{x} = 3 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (-4) \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \\ 10 \end{bmatrix}$$

$$\gamma: \vec{x} = 8 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + (-9) \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \\ 10 \end{bmatrix}$$

✓✓✓