Example

Diagonalize T where $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Characteristic equation : $\det(T - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \implies \lambda^2 + 1 = 0$$
$$\implies \lambda = \pm i$$

i - eigenspace

$$T\vec{x} = i\vec{x} \implies (T - \lambda I)\vec{x} = \vec{0}$$
$$\implies \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_1 = -i x_2$$

A Basis: $\left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} \right\}$

Check: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix}$
$= i\begin{bmatrix} 1 \\ i \end{bmatrix}$

$-i$ - eigenspace

$$(T - \lambda I)\vec{x} = \vec{0} \implies \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \quad x_1 = i x_2$$

A Basis: $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$

$\therefore$ A diagonalization of T is PDP^{-1} where $P = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$, $D = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$,

$P^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}$

$PDP^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
$= T$ ✓