Background Example

Find the general solution to the differential equation $\frac{dx}{dt} = k\,x$ and use that solution to help solve the system $\frac{d\vec{y}}{dt} = A\,\vec{y}$ given that $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $\vec{y}(0) = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$.

A $\underbrace{\frac{dx}{dt} = kx}$ $\Rightarrow \frac{1}{x}\frac{dx}{dt} = k \Rightarrow \frac{1}{x}dx = k\,dt$

$\Rightarrow \int \frac{1}{x}\,dx = \int k\,dt$

error 1 $\Rightarrow \ln(x) = kt + c$ x has to be positive

$\Rightarrow x = e^{kt+c}$

$\Rightarrow x = e^{kt}e^{c}$

$\Rightarrow \boxed{x = C_1 e^{kt}}$ B

error 2

C_1 can be "+" or "−" but $e^c > 0$ ∴ x can be negative again

Henceforth you may go straight from A to B

The actual Problem:

$\frac{d\vec{y}}{dt} = A\vec{y}$ where $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ + $\vec{y}(0) = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$

$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2y_1 \\ 3y_2 \end{bmatrix}$

$\Rightarrow \begin{cases} \frac{dy_1}{dt} = 2y_1 \\ \frac{dy_2}{dt} = 3y_2 \end{cases}$ A

$\Rightarrow \begin{cases} y_1 = C_1 e^{2t} \\ y_2 = C_2 e^{3t} \end{cases}$ B

$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \Rightarrow \begin{cases} -4 = C_1 e^0 \\ -2 = C_2 e^0 \end{cases}$

$\Rightarrow \begin{cases} C_1 = -4 \\ C_2 = -2 \end{cases}$

∴ The specific solution is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -4e^{2t} \\ -2e^{3t} \end{bmatrix}$

Check : $\frac{d}{dt}\begin{bmatrix} -4e^{2t} \\ -2e^{3t} \end{bmatrix} = \begin{bmatrix} -8e^{2t} \\ -6e^{3t} \end{bmatrix} = \begin{bmatrix} 2y_1 \\ 3y_2 \end{bmatrix}$ ✓

Example 2

Solve the system $\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = -y_1 + 4y_2 \end{cases}$ given that $y_1(0) = -5$ and $y_2(0) = 0$. Begin by writing the system in the form $\vec{y}' = A\,\vec{y}$ and "decoupling" the system by diagonalizing A and making substitutions based upon $\vec{y} = P\vec{w}$ and $\vec{y}' = P\vec{w}'$.

Mission 1: Decouple the System: $\begin{cases} y_1' = k_1 y_1 \\ y_2' = k_2 y_2 \end{cases}$

$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$; Define $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

eigenvalues of A: 2 and 3 (Anthony's TI-nSpire!)

2-eigenspace

$A\vec{x} = 2\vec{x} \implies \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Clearly $-x_1 + 2x_2 = 0$, so $x_1 = 2x_2$. Basis $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

3-eigenspace

$A\vec{x} = 3\vec{x} \implies \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x_1 = x_2$

Basis: $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$\therefore \quad A = PDP^{-1}$ where

$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$$A = PDP^{-1} \implies P^{-1}AP = P^{-1}PDP^{-1}P = D$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},\quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},\quad P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Define: $\vec{w} = P^{-1}\vec{y}$ so that $\vec{y} = P\vec{w}$ & $\vec{y}\,' = P\vec{w}\,'$

Given: $\vec{y}\,' = A\vec{y} \implies P\vec{w}\,' = A \cdot P\vec{w}$

$\implies P^{-1} \cdot P\vec{w}\,' = P^{-1} \cdot AP\vec{w}$

$\implies (P^{-1}P)\vec{w}\,' = (P^{-1}AP)\vec{w}$

$\implies I\vec{w}\,' = D\vec{w}$ ⇐ decouplage accomplished!

$\implies \vec{w}\,' = D\vec{w}$

Future: Go straight from α to ω

$$\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = -y_1 + 4y_2 \end{cases} \implies \begin{cases} w_1' = 2w_1 \\ w_2' = 3w_2 \end{cases}$$

$$\implies \begin{cases} w_1 = c_1 e^{2t} \\ w_2 = c_2 e^{3t} \end{cases}$$

$$\vec{y} = P\vec{w} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{3t} \end{bmatrix}$$

$$\implies \begin{cases} y_1 = 2c_1 e^{2t} + c_2 e^{3t} \\ y_2 = c_1 e^{2t} + c_2 e^{3t} \end{cases} \quad \text{(general solution)}$$

$$\vec{y}(0) = \begin{bmatrix} -5 \\ 0 \end{bmatrix} \implies \begin{cases} -5 = 2c_1 + c_2 \\ 0 = c_1 + c_2 \end{cases} \qquad (e^0 = 1)$$

$$\left[\begin{array}{cc|c} 2 & 1 & -5 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 5 \end{array}\right] \qquad \begin{matrix} c_1 = -5 \\ c_2 = 5 \end{matrix}$$

$\therefore$ The specific solution to the system of Diffy-Qs is:

$$\begin{cases} y_1 = -10e^{2t} + 5e^{3t} \\ y_2 = -5e^{2t} + 5e^{3t} \end{cases}$$

Check

$y_1' = -20e^{2t} + 15e^{3t}$

$y_1 + 2y_2 = (-10e^{2t} + 5e^{3t}) + 2(-5e^{2t} + 5e^{3t})$

$= -20e^{2t} + 15e^{3t}$

$= y_1'$ ✓

$y_2' = -10e^{2t} + 15e^{3t}$

$-y_1 + 4y_2 = -(-10e^{2t} + 5e^{3t}) + 4(-5e^{2t} + 5e^{3t})$

$= -10e^{2t} + 15e^{3t}$

$=$

Example 3

Solve the system $\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = 2y_1 + 4y_2 \end{cases}$ given that $y_1(0)=1$ and $y_2(0)=-1$. Begin by writing the system in the form $\vec{y}' = A\,\vec{y}$ and "decoupling" the system by diagonalizing A and making substitutions based upon $\vec{y} = P\vec{w}$ and $\vec{y}' = P\vec{w}'$.

$\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Characteristic Eq:

$\det(A - \lambda I) = 0 \implies \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$

$\implies (1-\lambda)(4-\lambda) - 4 = 0$

$\implies \lambda^2 - 5\lambda = 0$

$\implies \lambda(\lambda - 5) = 0$

eigenvalues: 0 and 5

0 - eigenspace

$\begin{cases} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{cases}$

By Inspection gen sol: $\begin{cases} x_1 = -2x_2 \\ x_2 \text{ is free} \end{cases}$ Basis: $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

5- eigenspace

$\begin{cases} x_1 + 2x_2 = 5x_1 \\ 2x_1 + 4x_2 = 5x_2 \end{cases}$

$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right]$

By Inspection gen sol: $\begin{cases} x_1 = \frac{1}{2}x_2 \\ x_2 \text{ is free} \end{cases}$ Basis: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$\therefore$ $A = PDP^{-1}$ where $P = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$, $P^{-1} = -\frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $P = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$, $P^{-1} = \frac{1}{5}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$, $\vec{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Define $\vec{w} = P^{-1}\vec{y}$ so that $\vec{y} = P\vec{w}$ and $\vec{y}\,' = P\vec{w}\,'$

$$\vec{y}\,' = A\vec{y} \implies \vec{w}\,' = D\vec{w}$$

$$\implies \begin{cases} w_1' = 0 \\ w_2' = 5w_2 \end{cases}$$

$$\implies \begin{cases} w_1 = C_1 \\ w_2 = C_2 e^{5t} \end{cases}$$

$$\vec{y} = P\vec{w} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 e^{5t} \end{bmatrix}$$

$$\implies \begin{cases} y_1 = -2C_1 + C_2 e^{5t} \\ y_2 = C_1 + 2C_2 e^{5t} \end{cases}$$

$$\vec{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \implies \begin{cases} -2C_1 + C_2 = 1 \\ C_1 + 2C_2 = -1 \end{cases}$$

$$\left[\begin{array}{cc|c} -2 & 1 & 1 \\ 1 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -3/5 \\ 0 & 1 & -1/5 \end{array}\right] \quad \begin{cases} C_1 = -3/5 \\ C_2 = -1/5 \end{cases}$$

$\therefore$ The specific solution to the system of Differential Equations is
$$\begin{cases} y_1 = \frac{6}{5} - \frac{1}{5}e^{5t} \\ y_2 = -\frac{3}{5} - \frac{2}{5}e^{5t} \end{cases}$$

Check

$y_1' = -e^{5t}$

$y_1 + 2y_2 = \left(\frac{6}{5} - \frac{1}{5}e^{5t}\right) + 2\left(\frac{-3}{5} - \frac{2}{5}e^{5t}\right)$
$= -e^{5t}$ ✓

$y_2' = -2e^{5t}$

$2y_1 + 4y_2$ ✓ Twice what I just checked.

$e^{\pi i} + 1 = 0$

Euler's Formula

$e^{\theta i} = \cos(\theta) + i\sin(\theta)$

MTH 261 – Mr. Simonds' class

Example 4

Solve the system $\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases}$ given that $y_1(0)=3$ and $y_2(0)=3$. Begin by writing the system in the form $\vec{y}' = A\,\vec{y}$ and "decoupling" the system by diagonalizing A and making substitutions based upon $\vec{y} = P\vec{w}$ and $\vec{y}' = P\vec{w}'$.

$\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

From Eigenvalue notes: $P = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $P^{-1} = \frac{1}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$

$(A = PDP^{-1})$

Define: $\vec{w} = P^{-1}\vec{y}$, so that $\vec{y} = P\vec{w}$ and $\vec{y}' = P\vec{w}'$

$$\vec{y}' = A\vec{y} \implies \vec{w}' = D\vec{w}$$

$$\implies \begin{cases} w_1' = i w_1 \\ w_2' = -i w_2 \end{cases}$$

$$\implies \begin{cases} w_1 = C_1 e^{it} \\ w_2 = C_2 e^{-it} \end{cases}$$

$$\implies \begin{cases} w_1 = C_1[\cos(t) + i\sin(t)] \\ w_2 = C_2[\cos(-t) + i\sin(-t)] \end{cases}$$

$$\implies \begin{cases} w_1 = C_1[\cos(t) + i\sin(t)] \\ w_2 = C_2[\cos(t) - i\sin(t)] \end{cases}$$

$$\vec{y} = P\vec{w} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \begin{bmatrix} C_1[\cos(t) + i\sin(t)] \\ C_2[\cos(t) - i\sin(t)] \end{bmatrix}$$

$$\implies \begin{cases} y_1 = C_1 i(\cos(t) + i\sin(t)) + C_2 i(\cos(t) - i\sin(t)) \\ y_2 = -C_1(\cos(t) + i\sin(t)) + C_2(\cos(t) - i\sin(t)) \end{cases}$$

$$\vec{y}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \Longrightarrow \begin{cases} 3 = C_1 i\,(\cos(0) + i\sin(0)) + C_2 i\,(\cos(0) - i\sin(0)) \\ 3 = -C_1(\cos(0) + i\sin(0)) + C_2(\cos(0) - i\sin(0)) \end{cases}$$

$$\Longrightarrow \begin{cases} iC_1 + iC_2 = 3 \\ -C_1 + C_2 = 3 \end{cases}$$

$$\left[\begin{array}{cc|c} i & i & 3 \\ -1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -3/2 \quad -3/2\, i \\ 0 & 1 & 3/2 \quad -3/2\, i \end{array}\right] \qquad \begin{cases} C_1 = -\frac{3}{2} - \frac{3}{2}i \\ C_2 = \frac{3}{2} - \frac{3}{2}i \end{cases}$$

$$\therefore \begin{cases} y_1 = \left(-\frac{3}{2} - \frac{3}{2}i\right)\cdot i\cdot(\cos(t) + i\sin(t)) + \left(\frac{3}{2} - \frac{3}{2}i\right)\cdot i\cdot(\cos(t) - i\sin(t)) \\ y_2 = \left(\frac{3}{2} + \frac{3}{2}i\right)(\cos(t) + i\sin(t)) + \left(\frac{3}{2} - \frac{3}{2}i\right)(\cos(t) - i\sin(t)) \end{cases}$$

$$\Longrightarrow \begin{cases} y_1 = \cancel{-\frac{3}{2}i\cos(t)} + \frac{3}{2}\sin(t) + \frac{3}{2}\cos(t) + \cancel{\frac{3}{2}i\sin(t)} \\ \qquad + \cancel{\frac{3}{2}i\cos(t)} + \frac{3}{2}\sin(t) + \frac{3}{2}\cos(t) - \cancel{\frac{3}{2}i\sin(t)} \\ y_2 = \frac{3}{2}\cos(t) + \cancel{\frac{3}{2}i\sin(t)} + \cancel{\frac{3}{2}i\cos(t)} - \frac{3}{2}\sin(t) \\ \qquad + \frac{3}{2}\cos(t) - \cancel{\frac{3}{2}i\sin(t)} - \cancel{\frac{3}{2}i\cos(t)} - \frac{3}{2}\sin(t) \end{cases}$$

$$\Longrightarrow \begin{cases} y_1 = 3\sin(t) + 3\cos(t) \\ y_2 = 3\cos(t) - 3\sin(t) \end{cases}$$

d'oh! $\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases}$