derivative notation $\quad x = f(t) \quad$ differential

$$\boxed{\frac{dx}{dt} = f'(t)}$$

$dx = f'(t)\, dt$

$dx = \frac{dx}{dt}\, dt$

Background Example

Find the general solution to the differential equation $\frac{dx}{dt} = k\,x$ and use that solution to help solve the system $\frac{d\vec{y}}{dt} = A\,\vec{y}$ given that $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $\vec{y}(0) = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$.

$$\frac{dx}{dt} = kx \implies \frac{1}{x}\frac{dx}{dt} = k \implies \int \frac{1}{x}\frac{dx}{dt}\,dt = \int k\,dt$$

$$\implies \int \frac{1}{x}\,dx = \int k\,dt$$

$$\implies \ln(x) = kt + C$$

$$\implies x = e^{kt+C} = e^{kt}\underbrace{e^{C}}_{C_1}$$

$$\therefore \boxed{x = C_1 e^{kt}} \quad \text{General Solution}$$

$$\frac{d\vec{y}}{dt} = A\vec{y} \implies \vec{y}\,' = A\vec{y}$$

$$\implies \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\implies \begin{cases} y_1' = 2y_1 \\ y_2' = 3y_2 \end{cases}$$

$$\implies \begin{cases} y_1 = C_1 e^{2t} \\ y_2 = C_2 e^{3t} \end{cases} \quad (\text{general solution})$$

$$\vec{y}(0) = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \implies \begin{cases} -4 = C_1 e^{0} \\ -2 = C_2 e^{0} \end{cases} \implies \begin{cases} C_1 = -4 \\ C_2 = -2 \end{cases}$$

$\therefore$ The specific solution is $\vec{y}(t) = \begin{bmatrix} -4e^{2t} \\ -2e^{3t} \end{bmatrix}$

Example 2

Solve the system $\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = -y_1 + 4y_2 \end{cases}$ given that $y_1(0) = -5$ and $y_2(0) = 0$. Begin by writing the system in the form $\vec{y}' = A\,\vec{y}$ and "decoupling" the system by diagonalizing A and making substitutions based upon $\vec{y} = P\vec{w}$ and $\vec{y}' = P\vec{w}'$.

$$\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = -y_1 + 4y_2 \end{cases} \Longrightarrow \vec{y}' = A\vec{y} \quad \text{where} \quad A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

We need to diagonalize A.

characteristic equation: $\det(A - \lambda I) = 0 \Longrightarrow \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = 0$

$$\Longrightarrow (1-\lambda)(4-\lambda) + 2 = 0$$
$$\Longrightarrow \lambda^2 - 5\lambda + 6 = 0$$
$$\Longrightarrow (\lambda - 3)(\lambda - 2) = 0$$
$$\Longrightarrow \lambda = 3 \text{ or } \lambda = 2$$

3 – eigen space

$(A - 3I)\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right]$ $x_1 = x_2$

Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

2 – eigenspace

$(A - 2I)\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right]$ $x_1 = 2x_2$

Basis: $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

$\therefore$ $A = PDP^{-1}$ where

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Back to diffy-Q...

. . . Define $\vec{w} = P^{-1}\vec{y}$. It follows that $\vec{y} = P\vec{w}$ and $\vec{y}\,' = P\vec{w}\,'$

So . . .

Recall: $A = PDP^{-1}$

So $P^{-1}AP = P^{-1}PDP^{-1}P = D$

$$\vec{y}\,' = A\vec{y} \implies P\vec{w}\,' = AP\vec{w}$$
$$\implies P^{-1}(P\vec{w}\,') = P^{-1}(AP\vec{w})$$
$$\implies (P^{-1}P)\vec{w}\,' = (P^{-1}AP)\vec{w}$$
$$\implies \vec{w}\,' = D\vec{w}$$

goto in the future

Decouple-age has been achieved.

$$\implies \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$
$$\implies \begin{cases} w_1' = 3w_1 \\ w_2' = 2w_2 \end{cases}$$
$$\implies \begin{cases} w_1 = c_1 e^{3t} \\ w_2 = c_2 e^{2t} \end{cases}$$

$\vec{y} = P\vec{w}$
$$\implies \vec{y} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{3t} \\ c_2 e^{2t} \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} + 2c_2 e^{2t} \\ c_1 e^{3t} + c_2 e^{2t} \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} -5 \\ 0 \end{bmatrix} \implies \begin{cases} c_1 + 2c_2 = -5 \\ c_1 + c_2 = 0 \end{cases} \implies \begin{cases} c_1 = 5 \\ c_2 = -5 \end{cases}$$

$\therefore$ The specific solution to the given system of differential equations is:

$$\vec{y}(t) = \begin{bmatrix} 5e^{3t} - 10e^{2t} \\ 5e^{3t} - 5e^{2t} \end{bmatrix}$$

Check

$$\begin{cases} y_1 = 5e^{3t} - 10e^{2t} \\ y_2 = 5e^{3t} - 5e^{2t} \end{cases} \Longrightarrow \begin{cases} y_1' = 15e^{3t} - 20e^{2t} \\ y_2' = 15e^{3t} - 10e^{2t} \end{cases}$$

Supposedly (from the given problem)

$$\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = -y_1 + 4y_2 \end{cases}$$

$$\begin{cases} y_1 + 2y_2 \\ -y_1 + 4y_2 \end{cases} \Longrightarrow \begin{cases} (5e^{3t} - 10e^{2t}) + 2(5e^{3t} - 5e^{2t}) \\ -(5e^{3t} - 10e^{2t}) + 4(5e^{3t} - 5e^{2t}) \end{cases}$$

$$\Longrightarrow \begin{cases} 15e^{3t} - 20e^{2t} \; \checkmark \\ 15e^{3t} - 10e^{2t} \; \checkmark \end{cases}$$

woohoo!

Example 3

Solve the system $\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = 2y_1 + 4y_2 \end{cases}$ given that $y_1(0)=1$ and $y_2(0)=-1$. Begin by writing the system in the form $\vec{y}\,' = A\,\vec{y}$ and "decoupling" the system by diagonalizing A and making substitutions based upon $\vec{y} = P\vec{w}$ and $\vec{y}\,' = P\vec{w}\,'$.

$\vec{y}\,' = A\vec{y}$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(4-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda = 0$$

$$\Rightarrow \lambda(\lambda - 5) = 0$$

Eigenvalues: 0 and 5

<u>0 - eigenspace</u>

$(A - 0I)\vec{x} = \vec{0}$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = -2x_2$ Basis $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

<u>5 - eigenspace</u>

$(A - 5I)\vec{x} = \vec{0}$

$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = \frac{1}{2}x_2$

Basis: $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

$A = PDP^{-1}$

$$P = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix},\ P^{-1} = -\frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}$$

Let $\vec{y} = P\vec{w}$, then

$$\vec{w}\,' = D\vec{w} \Rightarrow \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} w_1' = 0 \\ w_2' = 5w_2 \end{cases}$$

$$\Rightarrow \begin{cases} w_1 = C_1 \\ w_2 = C_2 e^{5t} \end{cases}$$

$$\vec{y} = P\vec{w} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 e^{5t} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2c_1 + c_2 e^{5t} \\ c_1 + 2c_2 e^{5t} \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} -2c_1 + c_2 = 1 \\ c_1 + 2c_2 = -1 \end{cases}$$

$$P\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ general

$$= \begin{bmatrix} -3/5 \\ -1/5 \end{bmatrix}$$

$\therefore$ The specific solution is $\begin{cases} y_1 = \frac{6}{5} - \frac{1}{5}e^{5t} \\ y_2 = -\frac{3}{5} - \frac{2}{5}e^{5t} \end{cases}$

Check $\begin{bmatrix} i \\ -1 \end{bmatrix}$

$A\begin{bmatrix} i \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} i \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -i \end{bmatrix} = i\begin{bmatrix} i \\ -1 \end{bmatrix}$ ✓

$y_1' = y_2 = 0\cdot y_1 + 1\cdot y_2$

$y_2' = -y_1 = -1\cdot y_1 + 0\cdot y_2$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Example 4

Solve the system $\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases}$ given that $y_1(0) = 3$ and $y_2(0) = 3$. Begin by writing the system in the form $\vec{y}' = A\,\vec{y}$ and "decoupling" the system by diagonalizing A and making substitutions based upon $\vec{y} = P\vec{w}$ and $\vec{y}' = P\vec{w}'$.

$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases} \Rightarrow \vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Characteristic equation: $\det(A - \lambda I) = 0$

$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$

i - eigenspace

$(A - iI)\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \xrightarrow{iR_1 + R_2 \to R_2} \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$-ix_1 + x_2 = 0 \Rightarrow x_2 = ix_1$

Basis: $\left\{ \begin{bmatrix} i \\ -1 \end{bmatrix} \right\}$

$-i$ - eigenspace

$x_1 = -\frac{1}{i}x_2$

$(A + iI)\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \xrightarrow{-iR_1 + R_2 \to R_2} \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$ix_1 + x_2 = 0$

$x_2 = -ix_1$

Basis: $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$

$\therefore\ A = PDP^{-1}$ where $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $P = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}$,

$P^{-1} = \frac{1}{2i}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$

$\frac{1}{i} = \frac{1}{i}\cdot\frac{i}{i} = \frac{i}{-1} = -i$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $P = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}$. $P^{-1} = \frac{1}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$

Define $\vec{w} = P^{-1}\vec{y}$ so that $\vec{y} = P\vec{w}$ and $\vec{y}\,' = P\vec{w}\,'$.

$$\vec{y}\,' = A\vec{y} \implies \vec{w}\,' = D\vec{w}$$

$$\implies \begin{cases} w_1' = i w_1 \\ w_2' = -i w_2 \end{cases}$$

$$\implies \begin{cases} w_1 = C_1 e^{it} \\ w_2 = C_2 e^{-it} \end{cases}$$

2x1

$$\vec{y} = P\vec{w} \implies \vec{y} = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{it} \\ c_2 e^{-it} \end{bmatrix}$$

$$\implies \begin{cases} y_1 = iC_1 e^{it} + iC_2 e^{-it} \\ y_2 = -C_1 e^{it} + C_2 e^{-it} \end{cases} \text{(general solution)}$$

$$\vec{y}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{cases} iC_1 + iC_2 = 3 \\ -C_1 + C_2 = 3 \end{cases}$$

$$\left[\begin{array}{cc|c} i & i & 3 \\ -1 & 1 & 3 \end{array}\right] \xrightarrow{-iR_1 + R_2 \to R_2} \left[\begin{array}{cc|c} i & i & 3 \\ 0 & 2 & 3-3i \end{array}\right]$$

$$2c_2 = 3 - 3i \implies c_2 = 1.5 - 1.5i$$

$$iC_1 + iC_2 = 3 \implies iC_1 + i(1.5 - 1.5i) = 3$$

$$\implies iC_1 + 1.5i + 1.5 = 3$$

$$\implies C_1 = \frac{1.5 - 1.5i}{i} \cdot \frac{-i}{-i} = -1.5 - 1.5i$$

Check: $\begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1.5 - 1.5i \\ 1.5 - 1.5i \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ ✓ ✓

iC_1 iC_2

$\therefore$ The specific solution is $\begin{cases} y_1 = (1.5 - 1.5i)e^{it} + (1.5 + 1.5i)e^{-it} \\ y_2 = (1.5 + 1.5i)e^{it} + (1.5 - 1.5i)e^{-it} \end{cases}$

$-C_1$ C_2

NORMAL STOPPING POINT.

Eulers Formula : $e^{ti} = \cos(t) + i\sin(t)$

We can rewrite our specific solution thus:

$$\begin{cases} y_1 = (1.5 - 1.5i)(\cos(t) + i\sin(t)) + (1.5 + 1.5i)(\cos(-t) + i\sin(-t)) \\ y_2 = (1.5 + 1.5i)(\cos(t) + i\sin(t)) + (1.5 - 1.5i)(\cos(-t) + i\sin(-t)) \end{cases}$$

$$\Longrightarrow \begin{cases} y_1 = (1.5 - 1.5i)(\cos(t) + i\sin(t)) + (1.5 + 1.5i)(\cos(t) - i\sin(t)) \\ y_2 = (1.5 + 1.5i)(\cos(t) + i\sin(t)) + (1.5 - 1.5i)(\cos(t) - i\sin(t)) \end{cases}$$

$$\Longrightarrow \begin{cases} y_1 = 1.5\cos(t) + \cancel{1.5\, i\sin(t)} - \cancel{1.5\, i\cos(t)} + 1.5\sin(t) \\ \qquad + 1.5\cos(t) - \cancel{1.5\, i\sin(t)} + \cancel{1.5\, i\cos(t)} + 1.5\sin(t) \\ y_2 = 1.5\cos(t) + \cancel{1.5\, i\sin(t)} + \cancel{1.5\, i\cos(t)} - 1.5\sin(t) \\ \qquad + 1.5\cos(t) - \cancel{1.5\, i\sin(t)} - \cancel{1.5\, i\cos(t)} - 1.5\sin(t) \end{cases}$$

$$\Longrightarrow \begin{cases} y_1 = 3\cos(t) + 3\sin(t) \\ y_2 = 3\cos(t) - 3\sin(t) \end{cases}$$

Check $\begin{cases} y_1' = y_2 \ ? \\ y_2' = -y_1 \end{cases}$ Woohoo!

$$\begin{cases} y_1' = -3\sin(t) + 3\cos(t) = y_2 \\ y_2' = -3\sin(t) - 3\cos(t) = -y_1 \end{cases}$$