$x = f(t) \longrightarrow dx = f'(t)\,dt$

$\frac{dx}{dt} = f'(t)$ $\quad = \frac{dx}{dt}\,dt$

Background Example

Find the general solution to the differential equation $\frac{dx}{dt} = k\,x$ and use that solution to help solve the system $\frac{d\vec{y}}{dt} = A\,\vec{y}$ given that $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $\vec{y}(0) = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$.

Background

$$\frac{dx}{dt} = kx \Longrightarrow \frac{1}{x}\frac{dx}{dt} = k$$
$$\Longrightarrow \int \frac{1}{x}\frac{dx}{dt}\,dt = \int k\,dt$$
$$\Longrightarrow \int \frac{1}{x}\,dx = \int k\,dt$$
$$\Longrightarrow \ln(x) = kt + C$$
$$\Longrightarrow x = e^{kt+C} = e^{kt}e^{C} = C_0 e^{kt}$$

$$\frac{d\vec{y}}{dt} = A\vec{y} \Longrightarrow \begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} \frac{dy_1}{dt} = 2y_1 \\ \frac{dy_2}{dt} = 3y_2 \end{cases} \Longrightarrow \begin{cases} y_1 = C_1 e^{2t} \\ y_2 = C_2 e^{3t} \end{cases}$$

$$\vec{y}(0) = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \Longrightarrow \begin{cases} -4 = C_1 e^0 \\ -2 = C_2 e^0 \end{cases} \Longrightarrow \begin{cases} C_1 = -4 \\ C_2 = -2 \end{cases}$$

$\therefore$ The specific solution is: $\vec{y}(t) = \begin{bmatrix} -4e^{2t} \\ -2e^{3t} \end{bmatrix}$

Example 2

Solve the system $\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = -y_1 + 4y_2 \end{cases}$ given that $y_1(0) = -5$ and $y_2(0) = 0$. Begin by writing the system in the form $\vec{y}\,' = A\,\vec{y}$ and "decoupling" the system by diagonalizing A and making substitutions based upon $\vec{y} = P\vec{w}$ and $\vec{y}\,' = P\vec{w}\,'$.

We have $\frac{d\vec{y}}{dt} = A\vec{y}$ where $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

$$\det(A - \lambda I) = 0 \implies \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$\implies (1-\lambda)(4-\lambda) + 2 = 0$$

$$\implies \lambda^2 - 5\lambda + 6 = 0$$

$$\implies (\lambda - 3)(\lambda - 2) = 0$$

$\therefore$ The eigenvalues are 2 and 3

2 – eigenspace

$(A - 2I)\vec{x} = \vec{0}$

$$\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = 2x_2 \implies$ Basis $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

3 – eigenspace

$(A - 3I)\vec{x} = \vec{0}$

$$\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = x_2 \implies$ Basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$\therefore$ $A = PDP^{-1}$ where $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

Define $\vec{y} = P\vec{w}$. Then, $\vec{y}\,' = P\vec{w}\,'$, $P^{-1}\vec{y} = \vec{w}$, and $P^{-1}\vec{y}\,' = \vec{w}\,'$

Attained knowledge

$$\vec{y}\,' = A\vec{y} \implies P\vec{w}\,' = AP\vec{w}$$

$$\implies P\vec{w}\,' = (PDP^{-1})P\vec{w}$$

$$\implies P^{-1}P\vec{w}\,' = P^{-1}(PDP^{-1})P\vec{w}$$

$$\implies I\vec{w}\,' = (P^{-1}P)D(P^{-1}P)\vec{w}$$

$$\implies \vec{w}\,' = D\vec{w}$$

Straight here

Because D is diagonal, the system has been decoupled.

$$\vec{w}' = D\vec{w} \Longrightarrow \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} w_1' = 2w_1 \\ w_2' = 3w_2 \end{cases}$$

$$\Longrightarrow \begin{cases} w_1 = c_1 e^{2t} \\ w_2 = c_2 e^{3t} \end{cases}$$

$$\Longrightarrow \vec{y} = P\vec{w} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{3t} \end{bmatrix} = \begin{bmatrix} 2c_1 e^{2t} + c_2 e^{3t} \\ c_1 e^{2t} + c_2 e^{3t} \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} -5 \\ 0 \end{bmatrix} \Longrightarrow \begin{cases} 2c_1 + c_2 = -5 \\ c_1 + c_2 = 0 \end{cases}$$

$$P\vec{c} = \begin{bmatrix} -5 \\ 0 \end{bmatrix} \Longrightarrow \vec{c} = P^{-1} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

$\therefore$ The specific solution is $\begin{cases} y_1 = 10e^{2t} - 5e^{3t} \\ y_2 = 5e^{2t} - 5e^{3t} \end{cases}$

Check $\quad \vec{y}' \stackrel{?}{=} \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \vec{y}$

$$\begin{bmatrix} 20e^{2t} - 15e^{3t} \\ 10e^{2t} - 15e^{3t} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 10e^{2t} - 5e^{3t} \\ 5e^{2t} - 5e^{3t} \end{bmatrix}$$

$$\begin{bmatrix} 20e^{2t} - 15e^{3t} \\ 10e^{2t} - 15e^{3t} \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 10e^{2t} - 5e^{3t} + 10e^{2t} - 10e^{3t} \\ -10e^{2t} + 5e^{3t} + 20e^{2t} - 20e^{3t} \end{bmatrix}$$

Example 3

Solve the system $\begin{cases} y_1' &= y_1 + 2y_2 \\ y_2' &= 2y_1 + 4y_2 \end{cases}$ given that $y_1(0)=1$ and $y_2(0)=-1$. Begin by writing the system in the form $\vec{y}' = A\,\vec{y}$ and "decoupling" the system by diagonalizing A and making substitutions based upon $\vec{y} = P\vec{w}$ and $\vec{y}' = P\vec{w}'$.

$\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(4-\lambda) - 4 = 0$

$\Rightarrow \lambda^2 - 5\lambda = 0$

$\Rightarrow \lambda(\lambda - 5) = 0$

Eigenvalues : 0 and 5

<u>0 - eigenspace</u>

$(A - 0I)\vec{x} = \vec{0}$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = -2x_2$ Basis $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

<u>5 - eigenspace</u>

$(A - 5I)\vec{x} = \vec{0}$

$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = \frac{1}{2}x_2$

Basis : $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

$A = PDP^{-1}$

$P = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix},\ P^{-1} = -\frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}$

Let $\vec{y} = P\vec{w}$, then

$$\vec{w}' = D\vec{w} \Rightarrow \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} w_1' = 0 \\ w_2' = 5w_2 \end{cases}$$

$$\Rightarrow \begin{cases} w_1 = C_1 \\ w_2 = C_2 e^{5t} \end{cases}$$

$$\vec{y} = P\vec{w} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 e^{5t} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2c_1 + c_2 e^{5t} \\ c_1 + 2c_2 e^{5t} \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} -2c_1 + c_2 = 1 \\ c_1 + 2c_2 = -1 \end{cases}$$

$$P\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ general

$$= \begin{bmatrix} -3/5 \\ -1/5 \end{bmatrix}$$

$\therefore$ The specific solution is $\begin{cases} y_1 = \frac{6}{5} - \frac{1}{5}e^{5t} \\ y_2 = -\frac{3}{5} - \frac{2}{5}e^{5t} \end{cases}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Example 4

Solve the system $\begin{cases} y_1' &= y_2 \\ y_2' &= -y_1 \end{cases}$ given that $y_1(0)=3$ and $y_2(0)=3$. Begin by writing the system in the form $\vec{y}'=A\,\vec{y}$ and "decoupling" the system by diagonalizing A and making substitutions based upon $\vec{y}=P\vec{w}$ and $\vec{y}'=P\vec{w}'$.

$\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1$$

$\therefore$ The eigenvalues are i and $-i$

i-eigenspace

$(A - iI)\vec{x} = \vec{0}$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$ $iR_1 + R_2 \to R_2$

$\begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}$ $x_2 = ix_1$

Basis $\left\{ \begin{bmatrix} i \\ -1 \end{bmatrix} \right\}$

$-i$-eigenspace

$(A + iI)\vec{x} = \vec{0}$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$ $-iR_1 + R_2 \to R_2$

$\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}$ $x_2 = -ix_1$

Basis: $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$

$\therefore A = PDP^{-1}$ where $P = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$,

$$P^{-1} = \frac{1}{2i}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

Let $\vec{y} = P\vec{w}$, then

$$\vec{w}' = D\vec{w} \implies \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\implies \begin{cases} w_1' = i w_1 \\ w_2' = -i w_2 \end{cases}$$

$$\implies \begin{cases} w_1 = c_1 e^{it} \\ w_2 = c_2 e^{-it} \end{cases}$$

$$\vec{y} = P\vec{w} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{it} \\ c_2 e^{-it} \end{bmatrix}$$

$$\implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 i e^{it} + c_2 i e^{-it} \\ -c_1 e^{it} + c_2 e^{-it} \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \implies \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} i c_1 + i c_2 \\ -c_1 + c_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} i & i & 3 \\ -1 & 1 & 3 \end{array}\right] \xrightarrow{-iR_1 + R_2 \to R_2} \left[\begin{array}{cc|c} i & i & 3 \\ 0 & 2 & 3 - 3i \end{array}\right]$$

$$c_2 = \frac{3}{2} - \frac{3}{2}i$$

$$c_1 = \frac{3 - i x_2}{i} = \left[3 - i\left(\frac{3}{2} - \frac{3}{2}i\right)\right] \cdot \frac{1}{i}$$

$$= \left[\frac{3}{2} - \frac{3}{2}i\right] \cdot \frac{1}{i} \cdot \frac{i}{i}$$

$$= \left(\frac{3}{2} - \frac{3}{2}i\right) \cdot \frac{i}{-1}$$

$$= -\frac{3}{2} - \frac{3}{2}i$$

$$\therefore \begin{cases} y_1 = \left(-\frac{3}{2} - \frac{3}{2}i\right) i e^{it} + \left(\frac{3}{2} - \frac{3}{2}i\right) i e^{-it} \\ y_2 = -\left(-\frac{3}{2} - \frac{3}{2}i\right) e^{it} + \left(\frac{3}{2} - \frac{3}{2}i\right) e^{-it} \end{cases}$$

so ...

$$\begin{cases} y_1 = -\frac{3}{2} i e^{it} + \frac{3}{2} e^{it} + \frac{3}{2} i e^{-it} + \frac{3}{2} e^{-it} \\ y_2 = \frac{3}{2} e^{it} + \frac{3}{2} i e^{it} + \frac{3}{2} e^{-it} - \frac{3}{2} i e^{-it} \end{cases}$$

<u>From Euler's Formula</u> $\qquad e^{bi} = \cos(b) + i\sin(b)$

$$e^{it} = \cos(t) + i\sin(t)$$

$$e^{-it} = \cos(-t) + i\sin(-t) = \cos(t) - i\sin(t)$$

$$\therefore \quad y_1 = -\frac{3}{2} i\left(\cos(t) + i\sin(t)\right) + \frac{3}{2}\left(\cos(t) + i\sin(t)\right) + \frac{3}{2} i\left(\cos(t) - i\sin(t)\right) + \frac{3}{2}\left(\cos(t) - i\sin(t)\right)$$

$$= \cancel{-\frac{3}{2} i\cos(t)} + \frac{3}{2}\sin(t) + \frac{3}{2}\cos(t) + \cancel{\frac{3}{2} i\sin(t)} + \cancel{\frac{3}{2} i\cos(t)} + \frac{3}{2}\sin(t) + \frac{3}{2}\cos(t) - \cancel{\frac{3}{2} i\sin(t)}$$

$$= 3\sin(t) + 3\cos(t)$$

$$y_2 = \frac{3}{2}\left(\cos(t) + i\sin(t)\right) + \frac{3}{2} i\left(\cos(t) + i\sin(t)\right) + \frac{3}{2}\left(\cos(t) - i\sin(t)\right) - \frac{3}{2} i\left(\cos(t) - i\sin(t)\right)$$

$$= \frac{3}{2}\cos(t) + \cancel{\frac{3}{2} i\sin(t)} + \cancel{\frac{3}{2} i\cos(t)} - \frac{3}{2}\sin(t) + \frac{3}{2}\cos(t) - \cancel{\frac{3}{2} i\sin(t)} - \cancel{\frac{3}{2} i\cos(t)} - \frac{3}{2}\sin(t)$$

$$= 3\cos(t) - 3\sin(t)$$

Behold: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3\sin(t) + 3\cos(t) \\ 3\cos(t) - 3\sin(t) \end{bmatrix}$

$$\Longrightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 3\cos(t) - 3\sin(t) \\ -3\sin(t) - 3\cos(t) \end{bmatrix}$$

$$= \begin{bmatrix} y_2 \\ -y_1 \end{bmatrix} \checkmark$$