$\ln(a) = b$

$a = e^b$

$x = f(t)$

$\frac{dx}{dt} = f'(t)$ $\quad dx = f'(t)\,dt$ (Definition) $= \frac{dx}{dt}\,dt$

Background Example

Find the general solution to the differential equation $\frac{dx}{dt} = k\,x$ and use that solution to help solve the system $\frac{d\vec{y}}{dt} = A\,\vec{y}$ given that $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $\vec{y}(0) = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$.

$\frac{dx}{dt} = kx \Rightarrow \frac{1}{x}\frac{dx}{dt} = k \Rightarrow \int \frac{1}{x}\frac{dx}{dt}\,dt = \int k\,dt$

you do not ever need to do this again

$\Rightarrow \int \frac{1}{x}\,dx = \int k\,dt \Rightarrow \ln(x) = kt + c$

$\Rightarrow x = e^{kt+c} \Rightarrow x = e^{kt}e^{c}$

$\Rightarrow x = C_1 e^{kt} \Leftarrow$ general solution

$\frac{d\vec{y}}{dt} = A\vec{y} \Rightarrow \begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\Rightarrow \begin{cases} dy_1/dt = 2y_1 \\ dy_2/dt = 3y_2 \end{cases}$

$\Rightarrow \begin{cases} y_1 = C_1 e^{2t} \\ y_2 = C_2 e^{3t} \end{cases}$ (assumed knowledge)

$\vec{y}(0) = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \Rightarrow \begin{cases} -4 = C_1 e^0 \\ -2 = C_2 e^0 \end{cases} \Rightarrow \begin{cases} C_1 = -4 \\ C_2 = -2 \end{cases}$

$\therefore$ The specific solution $\vec{y}(t) = \begin{bmatrix} -4e^{2t} \\ -2e^{3t} \end{bmatrix}$

Example 2

Solve the system $\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = -y_1 + 4y_2 \end{cases}$ given that $y_1(0) = -5$ and $y_2(0) = 0$. Begin by writing the system in the form $\vec{y}' = A\vec{y}$ and "decoupling" the system by diagonalizing A and making substitutions based upon $\vec{y} = P\vec{w}$ and $\vec{y}' = P\vec{w}'$.

$\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. The first step in decoupling the system is to diagonalize A. Let's do that!

We first need to find the eigenvalues.

Characteristic equation: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda) + 2 = 0$$
$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$
$$\Rightarrow \lambda = 3 \text{ or } \lambda = 2$$

2 - eigenspace

$$[A - \lambda I]\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x_1 = 2x_2$$

Basis: $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Check: $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ✓

3 - eigenspace

$$[A - \lambda I]\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x_1 = x_2$$

Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Check: $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$,

$P^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

Let $\begin{cases} \vec{y} = P\vec{w} \\ \vec{y}\,' = P\vec{w}\,' \end{cases}$ $\left(\text{note that} \begin{cases} P^{-1}\vec{y} = \vec{w} \\ P^{-1}\vec{y}\,' = \vec{w}\,' \end{cases} \right)$

$$\vec{y}\,' = A\vec{y} \implies P\vec{w}\,' = AP\vec{w}$$

Henceforth assumed knowledge

$$\implies P\vec{w}\,' = (PDP^{-1})P\vec{w}$$

$$\implies P^{-1}P\vec{w}\,' = P^{-1}(PDP^{-1})P\vec{w}$$

$$\implies I\vec{w}\,' = (P^{-1}P)D(P^{-1}P)\vec{w}$$

$$\implies \vec{w}\,' = D\vec{w}$$

The system is decoupled. Yahoo!

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \implies \begin{cases} w_1' = 3w_1 \\ w_2' = 2w_2 \end{cases}$$

$$\implies \begin{cases} w_1 = C_1 e^{3t} \\ w_2 = C_2 e^{2t} \end{cases}$$

$$\implies \vec{w} = \begin{bmatrix} C_1 e^{3t} \\ C_2 e^{2t} \end{bmatrix}$$

$$\implies P^{-1}\vec{y} = \begin{bmatrix} C_1 e^{3t} \\ C_2 e^{2t} \end{bmatrix}$$

$$\therefore \vec{y} = P\begin{bmatrix} C_1 e^{3t} \\ C_2 e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} C_1 e^{3t} \\ C_2 e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} C_1 e^{3t} + 2C_2 e^{2t} \\ C_1 e^{3t} + C_2 e^{2t} \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} -5 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} c_1 + 2c_2 = -5 \\ c_1 + c_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = 5 \\ c_2 = -5 \end{cases}$$

$\therefore$ The specific solution to the DE system is

$$\vec{y} = \begin{bmatrix} 5e^{3t} - 10e^{2t} \\ 5e^{3t} - 5e^{2t} \end{bmatrix}$$

Example 3

Solve the system $\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = 2y_1 + 4y_2 \end{cases}$ given that $y_1(0)=1$ and $y_2(0)=-1$. Begin by writing the system in the form $\vec{y}' = A\,\vec{y}$ and "decoupling" the system by diagonalizing A and making substitutions based upon $\vec{y} = P\,\vec{w}$ and $\vec{y}' = P\,\vec{w}'$.

$y' = A\vec{y}$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ before we we need to diagonalize A before we can decouple the system.

The eigenvalues of A are 0 and 5 (calc)

0 – eigenspace ($\lambda = 0$)

$(A - \lambda I)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$

$\Rightarrow x_1 = -2x_2$

Basis: $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ check: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

5 – eigenspace

$(A - \lambda I)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$

$\Rightarrow 2x_1 = x_2$

Basis: $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ check: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓

$\therefore A = PDP^{-1}$ where

$P = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$; $D = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$

$P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

Let $\begin{cases} \vec{y} = P\vec{w} \\ \vec{y}\,' = P\vec{w}\,' \end{cases}$ (note: $P^{-1}\vec{y} = \vec{w}$, $P^{-1}\vec{y}\,' = \vec{w}\,'$)

$$\vec{y}\,' = A\vec{y} \implies \vec{w}\,' = D\vec{w}$$

$$\implies \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\implies \begin{cases} w_1' = 5w_1 \\ w_2' = 0 \end{cases}$$

$$\implies \begin{cases} w_1 = C_1 e^{5t} \\ w_2 = C_2 \end{cases}$$

$$\implies \vec{w} = \begin{bmatrix} c_1 e^{5t} \\ c_2 \end{bmatrix}$$

$$\implies P^{-1}\vec{y} = \begin{bmatrix} c_1 e^{5t} \\ c_2 \end{bmatrix}$$

$$\therefore \vec{y} = P\begin{bmatrix} c_1 e^{5t} \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{5t} \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{5t} - 2c_2 \\ 2c_1 e^{5t} + c_2 \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \implies \begin{cases} c_1 - 2c_2 = 1 \\ 2c_1 + c_2 = -1 \end{cases}$$

$$\implies \begin{cases} c_1 = -1/5 \\ c_2 = -3/5 \end{cases}$$

$\therefore$ The specific solution to the DE system is:

$$\vec{y} = \begin{bmatrix} -\frac{1}{5}e^{5t} + \frac{6}{5} \\ -\frac{2}{5}e^{5t} - \frac{3}{5} \end{bmatrix}$$

Euler's Formula

$$e^{a+bi} = e^{a}[\cos(b) + i\sin(b)]$$

$$\begin{bmatrix} y_2 \\ -y_1 \end{bmatrix} = \begin{bmatrix} 0y_1 + 1y_2 \\ -1y_1 + 0y_2 \end{bmatrix}$$

Example 4

Solve the system $\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases}$ given that $y_1(0) = 3$ and $y_2(0) = 3$. Begin by writing the system in the form $\vec{y}' = A\,\vec{y}$ and "decoupling" the system by diagonalizing A and making substitutions based upon $\vec{y} = P\,\vec{w}$ and $\vec{y}' = P\,\vec{w}'$.

$\vec{y}' = A\vec{y}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

From previous notes

$A = PDP^{-1}$ where $P = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$, $D = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$,

$P^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}$

Let $\vec{y} = P\vec{w}$; then $\vec{y}' = P\vec{w}'$

and $\begin{cases} \vec{w} = P^{-1}\vec{y} \\ \vec{w}' = P^{-1}\vec{y}' \end{cases}$

Then $\vec{y}' = A\vec{y} \implies \vec{w}' = D\vec{w}$

$$\implies \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\implies \begin{cases} w_1' = -iw_1 \\ w_2' = iw_2 \end{cases}$$

$$\implies \begin{cases} w_1 = C_1e^{-it} \\ w_2 = C_2e^{it} \end{cases}$$

$$\implies \begin{cases} w_1 = C_1[\cos(-t) + i\sin(-t)] \\ w_2 = C_2[\cos(t) + i\sin(t)] \end{cases}$$

$$\implies \begin{cases} w_1 = C_1[\cos(t) - i\sin(t)] \\ w_2 = C_2[\cos(t) + i\sin(t)] \end{cases}$$

$$\vec{y} = P\,\vec{w} \Rightarrow \vec{y} = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \begin{bmatrix} c_1\cos(t) - i\,c_1\sin(t) \\ c_2\cos(t) + i\,c_2\sin(t) \end{bmatrix}$$

$$\Rightarrow y = \begin{bmatrix} i\,c_1\cos(t) + c_1\sin(t) + c_2\cos(t) + i\,c_2\sin(t) \\ c_1\cos(t) - i\,c_1\sin(t) + i\,c_2\cos(t) - c_2\sin(t) \end{bmatrix}$$

$$\Rightarrow y = \begin{bmatrix} (i\,c_1 + c_2)\cos(t) + (c_1 + i\,c_2)\sin(t) \\ (c_1 + i\,c_2)\cos(t) + (-i\,c_1 - c_2)\sin(t) \end{bmatrix}$$

$$y(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \Rightarrow \begin{cases} i\,c_1 + c_2 = 3 \\ c_1 + i\,c_2 = 3 \end{cases}$$

$$\left[\begin{array}{cc|c} i & 1 & 3 \\ 1 & i & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2}(3-3i) \\ 0 & 1 & \frac{1}{2}(3-3i) \end{array}\right] \Rightarrow \begin{cases} c_1 = \frac{1}{2}(3-3i) \\ c_2 = \frac{1}{2}(3-3i) \end{cases}$$

so...

$$y = \frac{1}{2}\begin{bmatrix} \left[(3i+3) + (3-3i)\right]\cos(t) + \left[(3-3i) + (3i+3)\right]\sin(t) \\ \left[(3-3i) + (3i+3)\right]\cos(t) + \left[(-3i-3) + (-3+3i)\right]\sin(t) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6\cos(t) + 6\sin(t) \\ 6\cos(t) - 6\sin(t) \end{bmatrix}$$

$\therefore$ The specific solution to our DE System is:

$$\vec{y}(t) = \begin{bmatrix} 3\cos(t) + 3\sin(t) \\ 3\cos(t) - 3\sin(t) \end{bmatrix}$$

Check: $y'(t) = \begin{bmatrix} -3\sin(t) + 3\cos(t) \\ -3\sin(t) - 3\cos(t) \end{bmatrix}$

$$= \begin{bmatrix} y_2(t) \\ -y_1(t) \end{bmatrix}$$