Vector Spaces that emerge from matrices (and affiliated vocabulary)

Suppose that A is and $n \times m$ matrix and that B is the reduced echelon equivalent of A. Then:

- The ***rank*** of A is the number of non-zero rows in B.
- The ***column space*** of A is the set of all linear combinations of the columns of A. The column space of A is a subspace of $\mathbb{R}^n$ and its dimension is equal to $\operatorname{rank}(A)$. The pivot columns of A form a basis for $\operatorname{col}(A)$. Basis comes from the original matrix
- The ***row space*** of A is the set of all linear combinations of the rows of A. The row space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $\operatorname{rank}(A)$. The non-zero rows of B for a basis for $\operatorname{row}(A)$ The basis comes from the row-reduced matrix
- The ***null space*** of A is the set of all solutions to the equation $A\vec{x} = \vec{0}$. The null space of A is a subspace of $\mathbb{R}^m$ and its dimension is equal to $m - \operatorname{rank}(A)$. One way to find a basis for $\operatorname{nul}(A)$ is to create vectors from the general solution to $A\vec{x} = \vec{0}$ where one vector is created for each free-variable by letting that free variable have a non-zero value whilst all the other free-variables are set to zero.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. State the correct number in each of the blanks below.

The rank of M is ___3___. (three non-zero rows)

The column space of M is a ___three___-dimensional subspace of $\mathbb{R}^{4}$.

The row space of M is a ___three___-dimensional subspace of $\mathbb{R}^{7}$.

The null space of M is a ___four___-dimensional subspace of $\mathbb{R}^{7}$.

$7 - 3$

\# of columns ↗ ↑ rank

(also # of free variables in solution to $M\vec{x} = \vec{0}$).

MTH 261 – Mr. Simonds' class

Note: rref(M) = M

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

State a basis for $\mathrm{row}(M)$. A basis for the rowspace of M is:

$\{[1,0,2,0,0,0,5], [0,1,-1,0,0,0,3], [0,0,0,1,0,5,-2]\}$

True or false? The stated basis for $\mathrm{row}(M)$ is also a basis for the row space of any matrix that is row equivalent to M. Justify your answer!

To create row equivalent matrices we can:

$R_i \leftrightarrow R_j$, $kR_i \to R_i$ $(k \neq 0)$, $kR_i + R_j \to R_j$

The first operation just reorders vectors and the next two of existing vectors, and hence cannot create linear combinations the original rows. manipulating rows increase the span of the linear indepence will have zero effect on basis. So, it is true that of our original basis is also a basis for any the given row equivelent matrix to M.

State a basis for $\mathrm{col}(M)$.

$X = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

True or false? The stated basis for $\mathrm{col}(M)$ is also a basis the column space for any matrix that is row equivalent to M. Justify your answer!

This is Sooooooo false

we can trivially create a non-zero entry in row 4 via row operations and that would take us outside of the span of X.

Example

Consider $M = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Answer each of the following questions about M.

State a basis for $\mathrm{nul}(M)$.

general solution to $M\vec{x} = \vec{0}$ is

$$\begin{cases} x_1 = -2x_3 - 5x_7 \\ x_2 = x_3 - 3x_7 \\ x_3 \text{ is free} \\ x_4 = -5x_6 + 2x_7 \\ x_5, x_6, x_7 \text{ are free} \end{cases}$$

A basis for nul(M) is:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

True or false? The stated basis for $\mathrm{nul}(M)$ is also a basis for the null space of any matrix that is row equivalent to M. Justify your answer!

This is Soooooooooo true. It's the entire basis of this class. That row manipulations result in equivalent linear systems (i.e. same solution set).

Example

Find bases for $\text{row}(A)$, $\text{col}(A)$, and $\text{nul}(A)$ where $A = \begin{bmatrix} 2 & -4 & -3 & 17 & 5 \\ -1 & 2 & 3 & -13 & -4 \\ 4 & -8 & 1 & 13 & 3 \end{bmatrix}$.

$$A \sim \begin{bmatrix} 1 & -2 & 0 & 4 & 1 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 2x_2 - 4x_4 - x_5$

Basis for row(A) is $\{[1, -2, 0, 4, 1], [0, 0, 1, -3, 1]\}$

Basis for col(A) is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix} \right\}$.

Basis for nul(A) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Example

Consider P_3, the set of all polynomials of degree three or less. Determine whether or not each of the following sets forms a basis for P_3. Justify each answer.

a. $\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2\}$ b. $\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2, t, -1+t^3\}$

c. $\{3-6t+t^3, 3t+7t^2-t^3, 1+2t+t^2, t\}$

P_3 is $\{c_0 + c_1 t + c_2 t^2 + c_3 t^3 \mid c_0, c_1, c_2, c_3 \in \mathbb{R}\}$

P_3 is isomorphic with $\mathbb{R}^4$. P_3 is four-dimensional.

Ergo, neither set (a) nor set (b) can form a basis for P_3 because they have the wrong number of vectors in them.

Let $T(\underbrace{c_0 + c_1 t + c_2 t^2 + c_3 t^3}_{\vec{p}(t)}) = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$

Let $A = \begin{bmatrix} 3 & 0 & 1 & 0 \\ -6 & 3 & 2 & 1 \\ 0 & 7 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$. Set (c) spans all of P_3 iff $Col(A)$ is all of $\mathbb{R}^4$. $Col(A)$ is all of $\mathbb{R}^4$ iff $\det(A) \neq 0$ (e-h-m triade from the green theorem)

$\det(A) = -4$

$\therefore$ Set (c) is a basis for P_3.

Example

Consider P_2, the set of all polynomials of degree two or less. Let V be the set of all polynomials in P_2 that satisfy the equation $\vec{p}(5)=0$. It is easily shown that V forms a subspace of P_2. Answer each of the following questions about V.

a. Find a basis for V and state the dimension of V.

b. Show that $g(x)=3x^2-34x+95\in V$.

c. Express g as a linear combination of the basis stated in part (a).

a. In order for $p(5)=0$, $(x-5)$ has to be a factor of p

So $p_1(x) = x-5 \in V$. But clearly, $p_1(x)$ does

not span V because you can't get a second

degree polynomial out of $p_1(x)$ and surely there

are second degree polynomials that satisfy $p(5)=0$.

To wit: $p_2(x) = (x-5)^2 = x^2-10x+25$.

Obviously $p_1(x), p_2(x)$ does span all of V because

P_2 is three-dimensional. V cannot be three-dimensional

because not every polynomial in P_2 is in V.

Example

The set of vectors of form $\begin{bmatrix} 2a & - & 3b & + & c \\ a & + & 3b & + & 5c \\ -3a & + & 2b & - & 4c \end{bmatrix}$ is easily shown to be a subspace of $\mathbb{R}^3$. Determine the dimension of this space.

$$\begin{bmatrix} 2a & -3b & +c \\ a & +3b & +5c \\ -3a & +2b & -4c \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + b\begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix} + c\begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix},$$ So the subspace is clearly spanned by $\left\{ \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix} \right\}$. We can write the vectors in the spanning set as the columns of a matrix to determine if the set is linearly independent.

$$\begin{vmatrix} 2 & -3 & 1 \\ 1 & 3 & 5 \\ -3 & 2 & -4 \end{vmatrix} = 0$$

Thus we know the three vectors are linearly <u>dependent</u>.

However, by inspection no two vectors in the set are linearly dependent.

$\therefore$ Any <u>pair</u> of vectors in the spanning set forms a basis for the given subspace of $\mathbb{R}^3$.