1. a. $\begin{bmatrix} 0 & 4 & 3 \\ 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \end{bmatrix}\begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 27 \\ 9 \\ 3/2 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix}$; ergo, the eigenvalue is $\frac{3}{2}$

   b. $\begin{bmatrix} 1 & -2 & 1 \\ -3 & -1 & -2 \\ -7 & 7 & -6 \end{bmatrix}\begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix}$; ergo, the eigenvalue is 0.

2.

a.

$$\begin{aligned} \det(A-\lambda I)=0 &\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0 \\ &\Rightarrow (1-\lambda)(3-\lambda)-8=0 \\ &\Rightarrow \lambda^2-4\lambda-5=0 \\ &\Rightarrow (\lambda-5)(\lambda+1)=0 \end{aligned}$$

$\therefore$ The eigenvalues of A are 5 and -1.

b.

$$\begin{aligned} \det(A-\lambda I)=0 &\Rightarrow \begin{vmatrix} -\lambda & 4 \\ -1 & 5-\lambda \end{vmatrix} = 0 \\ &\Rightarrow -\lambda(5-\lambda)+4=0 \\ &\Rightarrow \lambda^2-5\lambda+4=0 \\ &\Rightarrow (\lambda-4)(\lambda-1)=0 \end{aligned}$$

$\therefore$ The eigenvalues of A are 4 and 1.

3. a. For $\lambda = 3$:

$$\begin{aligned} (A-\lambda I)\vec{x}=\vec{0} &\Rightarrow \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow x_1 = 2x_2 \end{aligned}$$

$\therefore$ A basis for the 3-eigenspace is $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = -2$:

$$\begin{aligned} (A-\lambda I)\vec{x}=\vec{0} &\Rightarrow \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow x_1 = -\frac{1}{2}x_2 \end{aligned}$$

$\therefore$ A basis for the -2-eigenspace is $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.

b. For $\lambda = 1$:

$$(A-\lambda I)\vec{x} = \vec{0} \quad \Rightarrow \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 \text{ is free} \\ x_3 = 0 \end{cases}$$

$\therefore$ A basis for the 1-eigenspace is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 3$:

$$(A-\lambda I)\vec{x} = \vec{0} \quad \Rightarrow \begin{bmatrix} -2 & 0 & 2 \\ -1 & -2 & 1 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 2 \\ -1 & -2 & 1 \\ 2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = 0 \\ x_3 \text{ is free} \end{cases}$$

$\therefore$ A basis for the 3-eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = -1$:

$$(A-\lambda I)\vec{x} = \vec{0} \quad \Rightarrow \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases}$$

$\therefore$ A basis for the 3-eigenspace is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

4. $B = PDP^{-1}$ where $P = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$, and $P^{-1} = \frac{1}{6}\begin{bmatrix} -1 & -1 & 2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$

$$\begin{aligned} PDP^{-1} &= \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} -1 & -1 & 2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 0 & -6 & 12 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} -1 & -1 & 2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 42 & 6 & -12 \\ -18 & 18 & 36 \\ 12 & 12 & 12 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix} \quad \checkmark \end{aligned}$$

5. The characteristic equation is $\det(A - \lambda I) = 0$

$$\det(A-\lambda I) = 0 \;\Rightarrow\; \begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$
$$\Rightarrow\; (1-\lambda)(2-\lambda) = 0$$

$\therefore$ The eigenvalues of A are 1 and 2.

For $\lambda = 1$:

$$(A-\lambda I)\vec{x} = \vec{0} \;\Rightarrow\; \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow\; \begin{cases} x_1 \text{ is free} \\ x_2 = 0 \end{cases}$$

$\therefore$ A basis for the 1-eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 2$:

$$(A-\lambda I)\vec{x} = \vec{0} \;\Rightarrow\; \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow\; x_1 = x_2$$

$\therefore$ A basis for the 2-eigenspace is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Thus $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, and $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

So …

$$\begin{aligned} A^k &= PD^k P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2^k \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1+2^k \\ 0 & 2^k \end{bmatrix} \end{aligned}$$

This gives us …

$$\begin{aligned} A^{10} &= \begin{bmatrix} 1 & -1+2^{10} \\ 0 & 2^{10} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1023 \\ 0 & 1024 \end{bmatrix} \end{aligned}$$