

Eigenvalues and Eigenvectors (of square matrices)

A non-zero vector \vec{v} is called **an eigenvector** of the square matrix A if there exists a scalar, λ , with the property that $A\vec{v} = \lambda\vec{v}$. If such a vector and scalar exist, the scalar λ is called **an eigenvalue** of A .

The eigenvalues of A are the solutions to the equation $\det(A - \lambda I) = 0$; this equation is called **the characteristic equation** of A .

Example

Let's find the eigenvalues and eigenvectors of the matrix A where $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$.

Example

Whence the characteristic equation?

Example

Find the eigenvalues for $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ -1 & 12 & 3 & 0 \\ 4 & 4 & 2 & 0 \end{bmatrix}$.

Eigenspaces (of square matrices)

The set of all eigenvectors associated with the specific eigenvalue λ_i is called the

λ_i -eigenspace of A . The dimension of the λ_i -eigenspace is called the geometric multiplicity of λ_i .

Example

Let's find bases for the eigenspaces of B where $B = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

A Definition and a Theorem

The square matrices A and B are similar matrices if and only if there exists a matrix P with the property that $A = PBP^{-1}$ (or, similarly, $P^{-1}AP = B$). Similar matrices have the same characteristic equation.

NOTE: Not all matrices that share a characteristic equation are similar!

Diagonalization of an $n \times n$ matrix A

If A has n linearly independent eigenvectors, then A is similar to a diagonal matrix, D . Furthermore, $D = P^{-1}AP$ where the columns of P are composed of n linearly independent eigenvectors of A and the main diagonal entry in the i^{th} column of D is the eigenvalue that corresponds to the eigenvector in the i^{th} column of P . The product $PD P^{-1}$ is called a diagonalization of A .

Example

Find two distinct diagonalizations of A where $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$.

Example

Diagonalize B where $B = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$; use the result to simplify B^n where n is a natural number. To

get to the point more quickly, let's use our calculators to find the eigenvalues of M .

Example

What happens when we try to diagonalize M where $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$? To get to the point more expeditiously, let's use our calculators to find the eigenvalues of M .

Example

Consider the recursive series where $a_1 = 1$, $a_2 = 1$, and $a_k = 2a_{k-1} + 3a_{k-2}$ for $k \geq 3$. Let's find a general term formula (non-recursive) for a_k starting at $k = 3$.

Example

Diagonalize T where $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.