Eigenvalues and Eigenvectors (of square matrices)

A non-zero vector $\vec{v}$ is called ***an eigenvector*** of the square matrix A if there exists a scalar, λ, with the property that $A\vec{v} = \lambda\vec{v}$. If such a vector and scalar exist, the scalar λ is called ***an eigenvalue*** of A.

The eigenvalues of A are the solutions to the equation $\det(A - \lambda I) = 0$; this equation is called ***the characteristic equation*** of A.

Example

Let's find the eigenvalues and eigenvectors of the matrix A where $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$.

Characteristic equation

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(4-\lambda) - (-1) = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 9 = 0$$

$$\Rightarrow (\lambda - 3)^2 = 0$$

$$\Rightarrow \lambda = 3$$

3 - eigenspace

$$A\vec{x} = 3\vec{x} \Rightarrow \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2x_1 + x_2 = 3x_1 \\ -1x_1 + 4x_2 = 3x_2 \end{cases}$$

$$\Rightarrow \begin{cases} -1x_1 + x_2 = 0 \\ -1x_1 + x_2 = 0 \end{cases}$$

$\therefore\ x_1 = x_2$

$\therefore$ A basis for this eigenspace is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

notice

$$\underbrace{\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2-3

4-3

$$\begin{bmatrix} 2-3 & 1 \\ -1 & 4-3 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix}$$

Check: $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓

Example

Whence the characteristic equation?

$$A\vec{x} = \lambda\vec{x} \iff A\vec{x} = \lambda \cdot (I\vec{x})$$
$$\iff A\vec{x} = (\lambda I)\cdot\vec{x}$$
$$\iff A\vec{x} - (\lambda I)\cdot\vec{x} = \vec{0}$$
$$\iff (A - \lambda I)\vec{x} = \vec{0}$$

By definition, eigenvectors are non-zero vectors. So if $\vec{x}$ is an eigenvector, $(A-\lambda I)\vec{x} = \vec{0}$ has non trivial solutions. Thm 8, property m ... QED

$$\det(A - \lambda I) = 0$$

Example

Find the eigenvalues for $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ -1 & 12 & 3 & 0 \\ 4 & 4 & 2 & 0 \end{bmatrix}$.

Characteristic Equation

$$\det(A - \lambda I) = 0 \implies \begin{vmatrix} 3-\lambda & 0 & 0 & 0 \\ -4 & 6-\lambda & 0 & 0 \\ -1 & 12 & 3-\lambda & 0 \\ 4 & 4 & 2 & -\lambda \end{vmatrix} = 0$$

$$\implies -\lambda(3-\lambda)(6-\lambda)(3-\lambda) = 0$$

$\therefore$ The eigenvalues are 0, 3, and 6 with algebraic multiplicities, respectively, of 1, 2, and 1.

Eigenspaces (of square matrices)

The set of all eigenvectors associated with the specific eigenvalue λ_i is called the ***λ_i-eigenspace of A***. The dimension of the λ_i-eigenspace is called the geometric multiplicity of λ_i.

Example

Let's find bases for the eigenspaces of B where $B = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

Characteristic Equation

$$\det(B - \lambda I) = 0 \Rightarrow \begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)(-1)^2 \begin{vmatrix} 1-\lambda & 6 \\ -1 & 8-\lambda \end{vmatrix} + 2(-1)^3 \begin{vmatrix} -1 & 6 \\ -1 & 8-\lambda \end{vmatrix} + 2(-1)^4 \begin{vmatrix} -1 & 6 \\ 1-\lambda & 6 \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)[(1-\lambda)(8-\lambda)+6] - 2[-(8-\lambda)+6] + 2[-6-6(1-\lambda)] = 0$$

$$\Rightarrow (4-\lambda)(\lambda^2 - 9\lambda + 14) - 2(\lambda - 2) + 2(6\lambda - 12) = 0$$

$$\Rightarrow (4-\lambda)(\lambda-2)(\lambda-7) - 2(\lambda-2) + 12(\lambda-2) = 0$$

$$\Rightarrow (\lambda-2)[(4-\lambda)(\lambda-7) - 2 + 12] = 0$$

$$\Rightarrow (\lambda-2)(-\lambda^2 + 11\lambda - 18) = 0$$

$$\Rightarrow -(\lambda-2)(\lambda^2 - 11\lambda + 18) = 0$$

$$\Rightarrow -(\lambda-2)(\lambda-2)(\lambda-9) = 0$$

$\therefore$ The eigenvalues are 2 (algebraic multiplicity of 2) and 9 (algebraic multiplicity of 1)

2 - eigenspace

$B\vec{x} = 2\vec{x} \Rightarrow \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \vec{x} = \vec{0}$ (note: $2x_1 - x_2 + 6x_3 = 0$, ibid, ibid)

obviously, the general solution is $\begin{cases} x_1 = \frac{1}{2}x_2 - 3x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

$\therefore$ A basis for this eigenspace is $\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

note: a more utile basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

arbitrary

Mega check: $3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 4\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \\ -4 \end{bmatrix}$

$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} \begin{bmatrix} 15 \\ 6 \\ -4 \end{bmatrix} = \begin{bmatrix} 60-6-24 \\ 30+6-24 \\ 30-6-32 \end{bmatrix} = \begin{bmatrix} 30 \\ 12 \\ -8 \end{bmatrix} = 2\begin{bmatrix} 15 \\ 6 \\ -4 \end{bmatrix}$ ✓

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9 - eigenspace

$B\vec{x} = 9\vec{x} \Rightarrow \begin{bmatrix} -5 & -1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{bmatrix} \vec{x} = \vec{0}$

$\begin{bmatrix} -5 & -1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ general solution: $\begin{cases} x_1 = x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \end{cases}$

Bases: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Check: $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4-1+6 \\ 2+1+6 \\ 2-1+8 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}$

A Definition and a Theorem

The square matrices A and B are ***similar matrices*** if and only if there exists a matrix P with the property that $A = PBP^{-1}$ (or, similarly, $P^{-1}AP = B$). Similar matrices have the same characteristic equation.

NOTE: Not all matrices that share a characteristic equation are similar!

Diagonalization of an $n \times n$ matrix A

If A has n linearly independent eigenvectors, then A is similar to a diagonal matrix, D. Furthermore, $D = P^{-1}AP$ where the columns of P are composed of n linearly independent eigenvectors of A and the main diagonal entry in the i^{th} column of D is the eigenvalue that corresponds to the eigenvector in the i^{th} column of P. The product PDP^{-1} is called a ***diagonalization*** of A.

Example

Find two distinct diagonalizations of A where $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$.

Characteristic Equation

$A\vec{x} = \lambda\vec{x} \Rightarrow \det(A - \lambda I) = 0$

$\Rightarrow \begin{vmatrix} 4-\lambda & 2 & 3 \\ -1 & 1-\lambda & -3 \\ 2 & 4 & 9-\lambda \end{vmatrix} = 0$

$\Rightarrow (4-\lambda)(-1)^2 \begin{vmatrix} 1-\lambda & -3 \\ 4 & 9-\lambda \end{vmatrix} + 2(-1)^3 \begin{vmatrix} -1 & -3 \\ 2 & 9-\lambda \end{vmatrix} + 3(-1)^4 \begin{vmatrix} -1 & 1-\lambda \\ 2 & 4 \end{vmatrix} = 0$

$\Rightarrow (4-\lambda)[(1-\lambda)(9-\lambda)+12] - 2[-1(9-\lambda)+6] + 3[-4-2(1-\lambda)] = 0$

$\Rightarrow (4-\lambda)(\lambda^2 - 10\lambda + 21) - 2(\lambda - 3) + 3(2\lambda - 6) = 0$

$\Rightarrow (4-\lambda)(\lambda-3)(\lambda-7) - 2(\lambda-3) + 6(\lambda-3) = 0$

$\Rightarrow (\lambda-3)[(4-\lambda)(\lambda-7) - 2 + 6] = 0 \Rightarrow (\lambda-3)(-\lambda^2 + 11\lambda - 24) = 0$

$\Rightarrow -(\lambda-3)(\lambda-3)(\lambda-8) = 0$

$\therefore$ The eigenvalues are 3 (multiplicity 2) & 8 (algebraic multiplicity 1)

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$$

3- eigenspace

$$A\vec{x} = 3\vec{x} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

general solutions $\begin{cases} x_1 = -2x_2 - 3x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

Basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Check: $5\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - 7\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ -7 \end{bmatrix}$

$$\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}\begin{bmatrix} 11 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 33 \\ 15 \\ -21 \end{bmatrix} \checkmark$$

8 -eigen space

$$A\vec{x} = 8\vec{x} \Rightarrow \begin{bmatrix} -4 & 2 & 3 \\ -1 & -7 & -3 \\ 2 & 4 & 1 \end{bmatrix} \vec{x} = \vec{0}$$

$$\begin{bmatrix} -4 & 2 & 3 \\ -1 & -7 & -3 \\ 2 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{1}{2}x_3$

$x_2 = -\frac{1}{2}x_3$

Basis: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

check

$$\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \\ 16 \end{bmatrix} = 8\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \checkmark$$

Let's Diagonalize!

One option: $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & -2 & -3 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 7 & 3 \\ -2 & -4 & -1 \end{bmatrix}$

Check: $PDP^{-1} = \begin{bmatrix} 1 & -2 & -3 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}P^{-1} = \begin{bmatrix} 8 & -6 & -9 \\ -8 & 3 & 0 \\ 16 & 0 & 3 \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 7 & 3 \\ -2 & -4 & -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 20 & 10 & 15 \\ -5 & 5 & -15 \\ 10 & 20 & 45 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix} \checkmark$$

two option: $A = PDP^{-1}$ where $P = \begin{bmatrix} -3 & 1 & -2 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $P^{-1} = \frac{1}{5}\begin{bmatrix} -2 & -4 & -1 \\ 1 & 2 & 3 \\ 1 & 7 & 3 \end{bmatrix}$

Example

Diagonalize B where $B=\begin{bmatrix}3 & -1 & -2\\ 2 & 0 & -2\\ 2 & -1 & -1\end{bmatrix}$; use the result to simplify B^n where n is a natural number. To get to the point more quickly, let's use our calculators to find the eigenvalues of M.

The eigenvalues of B are $\lambda = 0$ (multiplicity 1) and $\lambda = 1$ (multiplicity 2)

0-eigenspace

$$B\vec{x} = 0\vec{x} \Rightarrow \begin{bmatrix}3 & -1 & -2\\ 2 & 0 & -2\\ 2 & -1 & -1\end{bmatrix}\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c}3 & -1 & -2 & 0\\ 2 & 0 & -2 & 0\\ 2 & -1 & -1 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \Rightarrow x_1 = x_3 \text{ \& } x_2 = x_3$$

Basis for 0-eigenspace: $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

1-eigenspace

$$B\vec{x} = 1\vec{x} \Rightarrow \begin{bmatrix}2 & -1 & -2\\ 2 & -1 & -2\\ 2 & -1 & -2\end{bmatrix}\vec{x} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow x_1 = \tfrac{1}{2}x_2 + x_3$$

Basis for 1-eigenspace is $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$

$\therefore$ A diagonalization for B is $B = PDP^{-1}$ where:

$$P = \begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 0\\ 1 & 0 & 1\end{bmatrix},\quad D = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\ \text{and}\ P^{-1} = \begin{bmatrix}-2 & 1 & 2\\ 1 & 0 & -1\\ 2 & -1 & -1\end{bmatrix}$$

Check this out: $B^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD\cdot I\cdot DP^{-1}$
$= PD^2P^{-1}$

Cut to the chase... $\Rightarrow$ $\vdots$

$$B^n = PD^nP^{-1}$$

So...

$$B^n = \begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}^n\begin{bmatrix}-2 & 1 & 2\\ 1 & 0 & -1\\ 2 & -1 & -1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}0^n & 0 & 0\\ 0 & 1^n & 0\\ 0 & 0 & 1^n\end{bmatrix}\begin{bmatrix}-2 & 1 & 2\\ 1 & 0 & -1\\ 2 & -1 & -1\end{bmatrix}$$

$$= \begin{bmatrix}0 & 1 & 1\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-2 & 1 & 2\\ 1 & 0 & -1\\ 2 & -1 & -1\end{bmatrix}$$

$$= \begin{bmatrix}3 & -1 & -2\\ 2 & 0 & -2\\ 2 & -1 & -1\end{bmatrix} \Leftarrow$$

Holy Moly, $B^n = B\ \forall n$

B is called <u>Idempotent</u>.

Example

What happens when we try to diagonalize M where $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$? To get to the point more expeditiously, let's use our calculators to find the eigenvalues of M.

The eigenvalues of M are 1(algebraic multiplicity 2) & 2(algebraic multiplicity 1)

1-eigenspace

$M\vec{x} = 1\cdot\vec{x} \Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so... $x_1 = x_3$ and $x_2 = x_3$

A basis for 1-eigenspace is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

In order to diagonalize M, we need to locate three linearly independent eigenvectors for M. Since the algebraic multiplicity of 1 is two, and alg mult $\geq$ geometric multiplicity, if said three vectors could be found, two would have to come from $\lambda = 1$. It's not to be.

Example

Consider the recursive thing where $a_1 = 1$, $a_2 = 1$, and $a_k = 2a_{k-1} + 3a_{k-2}$ for $k \geq 3$. Let's find a general term formula (non-recursive) for a_k starting at $k = 3$.

Observe: for $k \geq 3$, $\begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{k-1} \\ a_{k-2} \end{bmatrix}$. So...

$$\begin{bmatrix} a_3 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_4 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\vdots$

for $k \geq 3$, $\begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}^{k-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let's diagonalize! Define $G = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$

Calc eigenvalues: 3 and -1.

3-eigenspace

$G\vec{x} = 3\vec{x} \Rightarrow \begin{cases} 2x_1 + 3x_2 = 3x_1 \\ x_1 = 3x_2 \end{cases}$ ← Bingo! Basis $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

-1-eigenspace

$G\vec{x} = -1 \cdot \vec{x} \Rightarrow \begin{cases} 2x_1 + 3x_2 = -x_1 \\ x_1 = -x_2 \end{cases}$ ← Shazam! Basis: $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$\therefore$ $G = PDP^{-1}$ where $P = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$, $P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

$$\therefore \begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = G^{k-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= (PDP^{-1})^{k-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3^{k-2} & 0 \\ 0 & (-1)^{k-2} \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3^{k-1} & (-1)^{k-1} \\ 3^{k-2} & (-1)^{k-2} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\left[3^{k-1} + (-1)^{k-1}\right] \\ \frac{1}{2}\left[3^{k-2} + (-1)^{k-2}\right] \end{bmatrix}$$

$$\boxed{\therefore a_k = \frac{1}{2}\left[3^{k-1} + (-1)^{k-1}\right]}$$

Check:

$$a_{10} = \frac{1}{2}\left[3^9 + (-1)^9\right] = 9841$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad a+d=1$, $a^2+bc=a$

Example

Diagonalize T where $T=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. $\quad \begin{bmatrix} i \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -i \end{bmatrix} = i\begin{bmatrix} i \\ -1 \end{bmatrix}$

Characteristic equation

$$\det(T-\lambda I)=0 \implies \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix}=0$$
$$\implies \lambda^2+1=0$$
$$\implies \lambda=\pm i$$

i - eigenspace

$$T\vec{x}=i\vec{x} \implies \begin{cases} x_2 = ix_1 \\ -x_1 = ix_2 \end{cases} \qquad \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \xrightarrow{iR_1+R_2\to R_2} \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So ... $ix_1 = x_2$ $\quad$ Basis: $\left\{\begin{bmatrix} i \\ -1 \end{bmatrix}\right\}$ $\quad$ A different basis $\left\{\begin{bmatrix} 1 \\ i \end{bmatrix}\right\}$

$-i$ - eigenspace

$$T\vec{x}=-i\vec{x} \implies \begin{cases} x_2 = -ix_1 \\ -x_1 = -ix_2 \end{cases} \qquad \left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \xrightarrow{-iR_1+R_2} \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So ... $ix_1 = -x_2 \implies x_2 = -ix_1$ $\quad$ A basis: $\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$

$$\boxed{\therefore\ T = PDP^{-1} \text{ where } P=\begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix},\ D=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\ P^{-1}=\frac{1}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}}$$