Check

$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓

Eigenvalues and Eigenvectors (of square matrices)

A non-zero vector $\vec{v}$ is called ***an eigenvector*** of the square matrix A if there exists a scalar, λ, with the property that $A\vec{v} = \lambda\vec{v}$. If such a vector and scalar exist, the scalar λ is called ***an eigenvalue*** of A.

The eigenvalues of A are the solutions to the equation $\det(A - \lambda I) = 0$; this equation is called ***the characteristic equation*** of A.

Example

Let's find the eigenvalues and eigenvectors of the matrix A where $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$.

Characteristic equation:

$$\det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(4-\lambda) - (-1) = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 9 = 0$$

$$\Rightarrow (\lambda - 3)^2 = 0$$

The only Eigen value of A is 3 (algebraic multiplicity is 2)

3-Eigen Space ($\lambda = 3$)

There are non-trivial solutions to

$$A\vec{x} = 3\vec{x} \Rightarrow A\vec{x} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\vec{x}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\vec{x} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\vec{x} = \vec{0}$$

$$\Rightarrow \left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\vec{x} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 2-3 & 1 \\ -1 & 4-3 \end{bmatrix}\vec{x} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

upshot

$A\vec{x} = \lambda\vec{x}$ implies that $(A - \lambda I)\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Clearly $x_1 = x_2$

Solution vectors have form $x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is a basis for the 3-eigenspace.

The geometric multiplicity of $\lambda = 3$ is one (3-eigenspace is one-dimensional)

Example

Whence the characteristic equation?

$$A\vec{x} = \lambda\vec{x} \iff A\vec{x} = \lambda(I\vec{x})$$
$$\iff A\vec{x} = (\lambda I)\vec{x}$$
$$\iff A\vec{x} - \lambda I\vec{x} = \vec{0}$$
$$\iff (A - \lambda I)\vec{x} = \vec{0} \quad (\text{Eq } 1)$$

Eq 1 has non-trivial solutions iff $\det(A - \lambda I) = 0$.

Example

Find the eigenvalues for $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ -1 & 12 & 3 & 0 \\ 4 & 4 & 2 & 0 \end{bmatrix}$.

lower triangular

Characteristic eq: $\begin{vmatrix} 3-\lambda & 0 & 0 & 0 \\ -4 & 6-\lambda & 0 & 0 \\ -1 & 12 & 3-\lambda & 0 \\ 4 & 4 & 2 & -\lambda \end{vmatrix} = 0$

$$(3-\lambda)(6-\lambda)(3-\lambda)\cdot -\lambda = 0$$

Eigen values

3 (alg. mult. of 2)

6 (alg. mult of 1)

0 (alg. mult of 1)

$$\begin{array}{r|rrrr} 3 & 1 & -13 & 40 & -13 \\ & & 3 & -30 & 30 \\ \hline & 1 & -10 & 10 & \boxed{17} \end{array}$$

$3^3 - 13(3^2) + 40(3) - 13 = 17$

Eigenspaces (of square matrices)

The set of all eigenvectors associated with the specific eigenvalue λ_i is called the ***λ_i-eigenspace of A***. The dimension of the λ_i-eigenspace is called the geometric multiplicity of λ_i.

Example

Let's find bases for the eigenspaces of B where $B = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

Characteristic equation

$$\begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(-1)^2 \begin{vmatrix} 1-\lambda & 6 \\ -1 & 8-\lambda \end{vmatrix} + (-1)(-1)^3 \begin{vmatrix} 2 & 6 \\ 2 & 8-\lambda \end{vmatrix} + 6(-1)^4 \begin{vmatrix} 2 & 1-\lambda \\ 2 & -1 \end{vmatrix} = 0$$

$$(4-\lambda)\left[(1-\lambda)(8-\lambda) - (-6)\right] + \left[2(8-\lambda) - 12\right] + 6\left[-2 - 2(1-\lambda)\right] = 0$$

$$(4-\lambda)(\lambda^2 - 9\lambda + 14) + \left[-2\lambda + 4\right] + 6\left[2\lambda - 4\right] = 0$$

$$-\lambda^3 + 13\lambda^2 - 40\lambda + 36 = 0$$

$$\lambda^3 - 13\lambda^2 + 40\lambda - 36 = 0$$

$$\begin{array}{r|rrrr} 2 & 1 & -13 & 40 & -36 \\ & & 2 & -22 & 36 \\ \hline & 1 & -11 & 18 & \boxed{0} \end{array}$$

Rational Zero Thm.
Factor Thm.
Synthetic division

$$(\lambda - 2)(\lambda^2 - 11\lambda + 18) = 0$$

$$(\lambda - 2)(\lambda - 2)(\lambda - 9) = 0$$

The eigenvalues are 2 (alg. mult is 2) and 9 (alg. mult. is 1)

The eigenspace for 2 is either two or one dimensional

The eigenspace for 9 is definitely one dimensional.

(continued on next page)

2 - eigenspace

$A\vec{x} = 2\vec{x} \implies \begin{bmatrix} 4-2 & -1 & 6 \\ 2 & 1-2 & 6 \\ 2 & -1 & 8-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ obviously $x_1 = \frac{1}{2}x_2 - 3x_3$

Solution vectors have form

$x_2\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

A basis for this eigenspace is

$\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$. Another basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Check: $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ ✓

$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ ✓

9 - Eigenspace

$A\vec{x} = 9\vec{x} \implies \begin{bmatrix} 4-9 & -1 & 6 \\ 2 & 1-9 & 6 \\ 2 & -1 & 8-9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -5 & -1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Solution vectors have form

$\begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

A basis for the 9-eigenspace is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

check: $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix} = 9\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ✓

A Definition and a Theorem

The square matrices A and B are ***similar matrices*** if and only if there exists a matrix P with the property that $A = PBP^{-1}$ (or, similarly, $P^{-1}AP = B$). Similar matrices have the same characteristic equation.

NOTE: Not all matrices that share a characteristic equation are similar!

Diagonalization of an $n \times n$ matrix A

If A has n linearly independent eigenvectors, then A is similar to a diagonal matrix, D. Furthermore, $D = P^{-1}AP$ where the columns of P are composed of n linearly independent eigenvectors of A and the main diagonal entry in the i^{th} column of D is the eigenvalue that corresponds to the eigenvector in the i^{th} column of P. The product PDP^{-1} is called a ***diagonalization*** of A.

Example

Find two distinct diagonalizations of A where $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$.

The eigenvalues of A are 3 (alg. mult. is 2) and 8 (alg mult. is 1)

3-eigenspace

$A\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Solutions satisfy $x_1 = -2x_2 - 3x_3$

A basis for this eigenspace is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Check

$\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ ✓ $\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ 3 \end{bmatrix} = 3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ ✓

$(A - 8I)\vec{x} = \vec{0}$

8 - eigenspace

$A\vec{x} = 8\vec{x} \Rightarrow \begin{bmatrix} -4 & 2 & 3 \\ -1 & -7 & -3 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} -4 & 2 & 3 & 0 \\ -1 & -7 & -3 & 0 \\ 2 & 4 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Surely
$x_1 = \frac{1}{2}x_3$
$x_2 = -\frac{1}{2}x_3$

Basis: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

Check

$\begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \\ 16 \end{bmatrix} = 8\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ ✓

$\therefore \quad A = PDP^{-1}$ where

basis for 3-eigenspace

basis for 8-eigenspace

option 1// $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, $P = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, $P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 7 & 3 \\ -2 & -4 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

option 2// $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $P = \begin{bmatrix} -3 & 1 & -2 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $P^{-1} = \frac{1}{5}\begin{pmatrix} -2 & -4 & -1 \\ 1 & 2 & 3 \\ 1 & 7 & 3 \end{pmatrix}$

note
$$B^k = (PDP^{-1})(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) \quad k\text{-factors of } PDP^{-1}$$
$$= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)\cdots(P^{-1}P)DP^{-1}$$
$$= PD^kP^{-1} \quad (P^{-1}P = I \text{ goes away})$$

Example

Diagonalize B where $B = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$; use the result to simplify B^n where n is a natural number. To get to the point more quickly, let's use our calculators to find the eigenvalues of ~~M~~ B.

The eigenvalues are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$

1-eigenspace

$(B - 1\cdot I)\vec{x} = \vec{0}$ (analagous to $A\vec{x} = 1\cdot\vec{x}$)

$$\left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 2 & -1 & -2 & 0 \\ 2 & -1 & -2 & 0 \end{array}\right]$$

Gee golly, $x_1 = \frac{1}{2}x_2 + x_3$

Basis: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Check

$$\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

0-eigenspace

$(B - 0I)\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 3 & -1 & -2 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Clearly $x_1 = x_3$, $x_2 = x_3$

Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

check
$$\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

$\therefore B = PDP^{-1}$ where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$

$$B^k = PD^kP^{-1} = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^k P^{-1} = P\begin{bmatrix} 1^k & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & 0^k \end{bmatrix}P^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \leftarrow \text{That's frickin' B!}$$

$B^k = B \ \forall\, k \in \mathbb{Z}^+$

B is called idempotent

Example

What happens when we try to diagonalize M where $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$? To get to the point more expeditiously, let's use our calculators to find the eigenvalues of M.

The eigenvalues are $\lambda_1 = 1$ (mult 2), $\lambda_2 = 2$

1 – eigen space

$M\vec{x} = 1 \cdot \vec{x} \Rightarrow (M - 1 \cdot I)\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_1 = x_3 \\ x_2 = x_3 \end{array}$$

Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

The geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity.

$\lambda_1 = 1$ has geometric mult of 1

$\lambda_2 = 2$ cannot have any geometric multiplicity other than 1

$\therefore$ M does not have three linearly independent eigenvectors so it cannot be diagolized.

check $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} \checkmark$$

check $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 1 \end{bmatrix} \checkmark$$

Example

Consider the recursive series where $a_1 = 1$, $a_2 = 1$, and $a_k = 2a_{k-1} + 3a_{k-2}$ for $k \geq 3$. Let's find a general term formula (non-recursive) for a_k starting at $k = 3$.

note that

$$\begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_{k-1} \\ a_{k-2} \end{bmatrix}$$

Define $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ & diagonalize A.

$A\vec{x} = \lambda\vec{x} \Rightarrow (A - \lambda I)\vec{x} = \vec{0}$ This latter equation has non-trivial solutions iff $\underbrace{\det(A - \lambda I) = 0}_{\text{characteristic equation.}}$

$$\begin{vmatrix} 2-\lambda & 3 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(2-\lambda) - 3 = 0$$
$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda = 3 \text{ or } \lambda = -1$$

3 - eigenspace

$(A - 3I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 1 & -3 & 0 \end{array}\right]$$ Golly, $x_1 = 3x_2$

Basis: $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$

-1 - eigenspace

$(A - (-1)I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 1 & 1 & 0 \end{array}\right]$$ Gee, $x_1 = -x_2$

Basis: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So, $A = PDP^{-1}$ where $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$, $P = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$, $P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

$$A^k = (PDP^{-1})^k = PD^kP^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3^k & 0 \\ 0 & (-1)^k \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} 3^{k+1} & (-1)^{k+1} \\ 3^k & (-1)^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} 3^{k+1} + (-1)^{k+2} & 3^{k+1} + 3(-1)^{k+1} \\ 3^k + (-1)^{k+1} & 3^k + 3(-1)^k \end{bmatrix}$$

By observation (a_{10})

$$\boxed{a_k = \frac{1}{4}\left(2 \cdot 3^{k-1} + 3(-1)^{k-1} + (-1)^k\right)}$$

Let's find a_{10}

$$\begin{bmatrix} a_{10} \\ a_9 \end{bmatrix} = A^8\begin{bmatrix} a_2 \\ a_1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3^9 + (-1)^{10} & 3^9 + 3(-1)^9 \\ 3^8 + (-1)^9 & 3^8 + 3(-1)^8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore a_{10} = \frac{1}{4}\left(3^9 + (-1)^{10} + 3^9 + 3(-1)^9\right) = 9841$$

Example

Diagonalize T where $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

characteristic equation: $\det(T - \lambda I) = 0$ Shocking!

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$$

i - eigen space

$(T - iI)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Jeepers, $x_1 = -i x_2$

Basis: $\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$

$-i$ - eigen space

$(T + iI)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

by Jove, $x_1 = i x_2$

Basis: $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$

$\therefore$ $T = PDP^{-1}$ where

$$D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{i}{i} \cdot \frac{1}{-2i} \begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix}$$

$$= \frac{i}{2} \begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$$