Eigenvalues and Eigenvectors (of square matrices)

A non-zero vector $\vec{v}$ is called ***an eigenvector*** of the square matrix A if there exists a scalar, λ, with the property that $A\vec{v} = \lambda\vec{v}$. If such a vector and scalar exist, the scalar λ is called ***an eigenvalue*** of A.

The eigenvalues of A are the solutions to the equation $\det(A - \lambda I) = 0$; this equation is called ***the characteristic equation*** of A.

Example

Let's find the eigenvalues and eigenvectors of the matrix A where $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ and in the process demonstrate the logic behind the characteristic equation.

Characteristic Equation: $\det(A - \lambda I) = 0$

$\det(A-\lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)(4-\lambda) + 1 = 0$

$\Rightarrow \lambda^2 - 6\lambda + 9 = 0$

$\Rightarrow (\lambda - 3)^2 = 0$

$\therefore$ The only eigenvalue is 3

3- eigenspace

Solve: $(A - 3I)\vec{x} = \vec{0}$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Clearly $-x_1 + x_2 = 0$, so a basis for the 3-eigenspace is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

Logic

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad (A - \lambda I)\vec{x} = \vec{0}$$

This equation has non-trivial solutions iff $\det(A - \lambda I) = 0$.

Eigenspaces (of square matrices)

The set of all eigenvectors associated with the specific eigenvalue λ_i is called the ***λ_i-eigenspace of A***. The dimension of the λ_i-eigenspace is called the geometric multiplicity of λ_i.

Example

Let's find bases for the eigenspaces of B where $B = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

Characteristic Equation: $\det(B - \lambda I) = 0$

$$\det(B-\lambda I) = 0 \implies \begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix} = 0$$

$$\implies (4-\lambda)(-1)^2 \begin{vmatrix} 1-\lambda & 6 \\ -1 & 8-\lambda \end{vmatrix} + 2(-1)^3 \begin{vmatrix} -1 & 6 \\ -1 & 8-\lambda \end{vmatrix} + 2(-1)^4 \begin{vmatrix} -1 & 6 \\ 1-\lambda & 6 \end{vmatrix} = 0$$

$$\implies (4-\lambda)\left[(1-\lambda)(8-\lambda)+6\right] - 2\left[-(8-\lambda)+6\right] + 2\left[-6-6(1-\lambda)\right] = 0$$

$$\implies (4-\lambda)(\lambda^2 - 9\lambda + 14) - 2(\lambda - 2) + 2(6\lambda - 12) = 0$$

$$\implies \underline{(4-\lambda)(\lambda-7)}(\lambda-2) - 2(\lambda-2) + 12(\lambda-2) = 0$$

$$\implies \left[\underline{(-\lambda^2 + 11\lambda - 28)} - 2 + 12\right](\lambda - 2) = 0$$

$$\implies (-\lambda^2 + 11\lambda - 18)(\lambda - 2) = 0$$

$$\implies -(\lambda^2 - 11\lambda + 18)(\lambda - 2) = 0$$

$$\implies -(\lambda - 2)(\lambda - 9)(\lambda - 2) = 0$$

$\therefore$ The Eigen values are 2 and 9.

2- eigenspace

$(B-2I)\vec{x}=\vec{0}$

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly $2x_1 - x_2 + 6x_3 = 0$

So $x_1 = \frac{1}{2}x_2 - 3x_3$

∴ A basis for the 2-eigenspace is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Geometric multiplicity is 2

[The algebraic multiplicity of $\lambda = 2$ is also 2 because the characteristic equation simplified to $(\lambda-2)^2(\lambda-9)=0$. It's always the case that geometric multiplicity ≤ algebraic multiplicity)

9- eigenspace

$(B-9I)\vec{x}=\vec{0}$

$$\begin{bmatrix} -5 & -1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & -1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly $x_1 = x_3$ and $x_2 = x_3$

So a basis for the 9-eigenspace is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$\therefore$ The eigenvalues are 3 (A.M. is 2) and 8 (A.M. is 1)

3 - eigenspace

$(A - 3I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly $x_1 = -2x_2 - 3x_3$

Basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

8 - eigenspace

$(A - 8I)\vec{x} = \vec{0}$

$$\begin{bmatrix} -4 & 2 & 3 \\ -1 & -7 & -3 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 & 3 \\ -1 & -7 & -3 \\ 2 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

$A = PDP^{-1}$ where:

option 1

$$P = \begin{bmatrix} 1 & -2 & -3 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 7 & 3 \\ -2 & -4 & -1 \end{bmatrix}$$

option 2

$$P = \begin{bmatrix} -3 & 1 & -2 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{5}\begin{bmatrix} -2 & -4 & -1 \\ 1 & 2 & 3 \\ 1 & 7 & 3 \end{bmatrix}$$

using option 1:
$$\begin{aligned} A^3 &= PD\underline{P^{-1} \cdot P}D\underline{P^{-1} \cdot P}DP^{-1} \\ &= PD \cdot \underline{I} \cdot D \cdot \underline{I} \cdot D \cdot P^{-1} \\ &= PD^3P^{-1} \\ &= \begin{bmatrix} 1 & -2 & -3 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 8^3 & 0 & 0 \\ 0 & 3^3 & 0 \\ 0 & 0 & 3^3 \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 7 & 3 \\ -2 & -4 & -1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 1 & -2 & -3 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8^3 & 0 & 0 \\ 0 & 3^3 & 0 \\ 0 & 0 & 3^3 \end{bmatrix} \cdot \frac{1}{5} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 7 & 3 \\ -2 & -4 & -1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 8^3 & -2\cdot 3^3 & -3\cdot 3^3 \\ -8^3 & 3^3 & 0 \\ 2\cdot 8^3 & 0 & 3^3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 7 & 3 \\ -2 & -4 & -1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 8^3+4\cdot 3^3 & 2\cdot 8^3-2\cdot 3^3 & 3\cdot 8^3-3\cdot 3^3 \\ -8^3+3^3 & -2\cdot 8^3+7\cdot 3^3 & -3\cdot 8^3+3\cdot 3^3 \\ 2\cdot 8^3-2\cdot 3^3 & 4\cdot 8^3-4\cdot 3^3 & 6\cdot 8^3-3^3 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 620 & 970 & 1455 \\ -485 & -835 & -1455 \\ 970 & 1940 & 3045 \end{bmatrix}$$

$$= \begin{bmatrix} 124 & 194 & 291 \\ -97 & -167 & -291 \\ 194 & 388 & 609 \end{bmatrix}$$

A Definition and a Theorem

The square matrices A and B are ***similar matrices*** if and only if there exists a matrix P with the property that $A = PBP^{-1}$ (or, similarly, $P^{-1}AP = B$). Similar matrices have the same characteristic equation.

NOTE: Not all matrices that share a characteristic equation are similar!

They have the same Eigenvalues

Diagonalization of an $n \times n$ matrix A

If A has n linearly independent eigenvectors, then A is similar to a diagonal matrix, D. Furthermore, $D = P^{-1}AP$ where the columns of P are composed of n linearly independent eigenvectors of A and the main diagonal entry in the i^{th} column of D is the eigenvalue that corresponds to the eigenvector in the i^{th} column of P. The product PDP^{-1} is called a ***diagonalization*** of A.

Example

Find two distinct diagonalizations of A where $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$; then use one of the diagonalizations to determine A^3.

Characteristic Equation: $\det(A - \lambda I) = 0$

$$\det(A-\lambda I) = 0 \Rightarrow \begin{vmatrix} 4-\lambda & 2 & 3 \\ -1 & 1-\lambda & -3 \\ 2 & 4 & 9-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)(-1)^2 \begin{vmatrix} 1-\lambda & -3 \\ 4 & 9-\lambda \end{vmatrix} + (2)(-1)^3 \begin{vmatrix} -1 & -3 \\ 2 & 9-\lambda \end{vmatrix} + (3)(-1)^4 \begin{vmatrix} -1 & 1-\lambda \\ 2 & 4 \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)[(1-\lambda)(9-\lambda)+12] - 2[-(9-\lambda)+6] + 3[-4-2(1-\lambda)] = 0$$

$$\Rightarrow (4-\lambda)(\lambda^2 - 10\lambda + 21) - 2(\lambda - 3) + 3(2\lambda - 6) = 0$$

$$\Rightarrow (4-\lambda)(\lambda - 7)(\lambda - 3) - 2(\lambda - 3) + 6(\lambda - 3) = 0$$

$$\Rightarrow [(-\lambda^2 + 11\lambda - 28) - 2 + 6](\lambda - 3) = 0$$

$$\Rightarrow -(\lambda^2 - 11\lambda + 24)(\lambda - 3) = 0$$

$$\Rightarrow -(\lambda - 3)(\lambda - 8)(\lambda - 3) = 0$$

Example

Diagonalize B where $B=\begin{bmatrix}3&-1&-2\\2&0&-2\\2&-1&-1\end{bmatrix}$; use the result to simplify B^n where n is a natural number.

from the calculator, the eigenvalues are 1 (a.m. of 2) and 0 (a.m. of 1)

1 - eigenspace

$(B-I)\vec{x}=\vec{0}$

$$\begin{bmatrix}2&-1&-2\\2&-1&-2\\2&-1&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

clearly $x_1=\frac{1}{2}x_2+x_3$

Basis: $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$

0 - eigenspace

$(B-0I)\vec{x}=\vec{0}$

$$\begin{bmatrix}3&-1&-2\\2&0&-2\\2&-1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}3&-1&-2\\2&0&-2\\2&-1&-1\end{bmatrix}\sim\begin{bmatrix}1&0&-1\\0&1&-1\\0&0&0\end{bmatrix}\begin{matrix}x_1=x_3\\x_2=x_3\end{matrix}$$

Basis: $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$.

$\therefore\quad B=PDP^{-1}$

where $P=\begin{bmatrix}1&1&1\\2&0&1\\0&1&1\end{bmatrix}$, $D=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$, $P^{-1}=\begin{bmatrix}1&0&-1\\2&-1&-1\\-2&1&2\end{bmatrix}$

$$\therefore\quad B^k=PD^kP^{-1}$$

$$=\begin{bmatrix}1&1&1\\2&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1^k&0&0\\0&1^k&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&0&-1\\2&-1&-1\\-2&1&2\end{bmatrix}$$

$$=\begin{bmatrix}1&1&0\\2&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&-1\\2&-1&-1\\-2&1&2\end{bmatrix}$$

$$=\begin{bmatrix}3&-1&-2\\2&0&-2\\2&-1&-1\end{bmatrix}$$

$$=B$$

BAM! $B^k=B\ \forall k$

B is called idempotent.

Example

What happens when we try to diagonalize M where $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$?

from the calculator, the eigenvalues are
2 (algebraic multiplicity 1) & 1 (alge. mult. is 2)

1 - eigenspace

$(M - I)\vec{x} = \vec{0}$

$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = x_3$, $x_2 = x_3$

Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

If one of the eigenvalues had a geometric multiplicity of 2, it would have been this one. It didn't. We cannot come up with three linearly independent eigenvectors.
$\therefore$ The matrix is not diagonalizable.

Example

Find the eigenvalues for each matrix.

$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ -1 & 12 & 3 & 0 \\ 4 & 4 & 2 & 0 \end{bmatrix}$

The eigenvalues are 3, 6, 3, and 0
lower triangular matrix, so $\det(A - \lambda I)$ is just
$(3-\lambda)(6-\lambda)(3-\lambda)(0-\lambda)$

$B = \begin{bmatrix} -1.9 & 14.4 & -8.4 & 34.8 \\ 1.6 & -2.7 & 3.2 & -1.6 \\ 1.2 & -8.0 & 4.7 & -18.2 \\ 1.6 & -1.6 & 3.2 & -2.7 \end{bmatrix}$

2.3, -1.1, -1.1, 2.7
from calculator

Example

Diagonalize T where $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Characteristic Equation $\det(T - \lambda I) = 0$

$$\det(T - \lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i$$

i - eigenspace

$(T - iI)\vec{x} = \vec{0}$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-ix_1 + x_2 = 0 \quad -x_1 - ix_2 = 0$

$x_2 = ix_1 \quad x_1 = -ix_2$

Basis: $\left\{ \begin{bmatrix} i \\ -1 \end{bmatrix} \right\}$

$-i$ - eigenspace

$(T + iI)\vec{x} = \vec{0}$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$ix_1 = -x_2 \quad x_1 = ix_2$

Basis: $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$

$\therefore \quad T = PDP^{-1}$, where

$$P = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad P^{-1} = \frac{1}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$$

Check: $\begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -i & -i \end{bmatrix}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = T \checkmark$$