$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x_1 + x_2 \\ -x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 - 3x_1 + x_2 \\ -x_1 + 4x_2 - 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-3 & 1 \\ -1 & 4-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$A - \lambda I$ where $\lambda = 3$

Eigenvalues and Eigenvectors (of square matrices)

A non-zero vector $\vec{v}$ is called ***an eigenvector*** of the square matrix A if there exists a scalar, λ, with the property that $A\vec{v} = \lambda\vec{v}$. If such a vector and scalar exist, the scalar λ is called ***an eigenvalue*** of A.

The eigenvectors are the solution to $(A - \lambda I)\vec{x} = \vec{0}$

The eigenvalues of A are the solutions to the equation $\det(A - \lambda I) = 0$; this equation is called ***the characteristic equation*** of A.

Example

Let's find the eigenvalues and eigenvectors of the matrix A where $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ and in the process demonstrate the logic behind the characteristic equation.

If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of A, then:

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$$

$$A\vec{x} = I \cdot \lambda\vec{x} \Rightarrow A\vec{x} = \lambda I\vec{x}$$

$$\Rightarrow A\vec{x} - \lambda I\vec{x} = \vec{0}$$

$$\Rightarrow (A - \lambda I)\vec{x} = \vec{0}$$

$\vec{x} \neq \vec{0} \Rightarrow (A - \lambda I)\vec{x} = \vec{0}$ has nontrivial solutions

$\Rightarrow A - \lambda I$ is not invertable

$\Rightarrow \det(A - \lambda I) = 0$

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doing part $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix}$

$$\det(A - \lambda I) = 0 \Rightarrow (2-\lambda)(4-\lambda) + 1 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 9 = 0$$

$$\Rightarrow (\lambda - 3)^2 = 0$$

$\therefore$ The only eigenvalue is 3.

Eigenspaces (of square matrices)

The set of all eigenvectors associated with the specific eigenvalue λ_i is called the ***λ_i-eigenspace of A***. The dimension of the λ_i-eigenspace is called the geometric multiplicity of λ_i.

Example

Let's find bases for the eigenspaces of B where $B = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

Characteristic Equation: $\det(B - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(-1)^{1+1}\begin{vmatrix} 1-\lambda & 6 \\ -1 & 8-\lambda \end{vmatrix} + (-1)(-1)^{1+2}\begin{vmatrix} 2 & 6 \\ 2 & 8-\lambda \end{vmatrix} + (6)(-1)^{1+3}\begin{vmatrix} 2 & 1-\lambda \\ 2 & -1 \end{vmatrix} = 0$$

$$(4-\lambda)(\lambda^2 - 9\lambda + 14) + (-2\lambda + 4) + 6(2\lambda - 4) = 0$$

$$(4-\lambda)(\lambda - 7)(\lambda - 2) + (-2)(\lambda - 2) + 12(\lambda - 2) = 0$$

$$(\lambda - 2)\left[(4-\lambda)(\lambda - 7) + (-2) + 12\right] = 0$$

$$(\lambda - 2)\left[-\lambda^2 + 11\lambda - 18\right] = 0$$

$$(\lambda - 2) \cdot -1(\lambda^2 - 11\lambda + 18) = 0$$

$$-(\lambda - 2)(\lambda - 9)(\lambda - 2) = 0$$

$\therefore$ The Eigenvalues are 2 and 9

2 - eigenspace

$(B-\lambda I)\vec{x}=\vec{0}$

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

obviously $2x_1 - x_2 + 6x_3 = 0$ so $x_1 = \frac{1}{2}x_2 - 3x_3$

vectors in this eigenspace have form:

$$\begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Check

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

9 - eigenspace

$(B-\lambda I)\vec{x}=\vec{0}$

$$\begin{bmatrix} -5 & -1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -5 & -1 & 6 & 0 \\ 2 & -8 & 6 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

General solution: $\begin{cases} x_1 = x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \end{cases}$

9 - eigenvectors: $x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

check: $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}$ ✓

A Definition and a Theorem

The square matrices A and B are ***similar matrices*** if and only if there exists a matrix P with the property that $A = PBP^{-1}$ (or, similarly, $P^{-1}AP = B$). Similar matrices have the same characteristic equation.

NOTE: Not all matrices that share a characteristic equation are similar!

Diagonalization of an $n \times n$ matrix A

If A has n linearly independent eigenvectors, then A is similar to a diagonal matrix, D. Furthermore, $D = P^{-1}AP$ where the columns of P are composed of n linearly independent eigenvectors of A and the main diagonal entry in the i^{th} column of D is the eigenvalue that corresponds to the eigenvector in the i^{th} column of P. The product PDP^{-1} is called a ***diagonalization*** of A.

Example

Find two distinct diagonalizations of A where $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$; then use one of the diagonalizations to determine A^3.

Characteristic equation: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & 2 & 3 \\ -1 & 1-\lambda & -3 \\ 2 & 4 & 9-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(1)\begin{vmatrix} 1-\lambda & -3 \\ 4 & 9-\lambda \end{vmatrix} + (2)(-1)\begin{vmatrix} -1 & -3 \\ 2 & 9-\lambda \end{vmatrix} + (3)(1)\begin{vmatrix} -1 & 1-\lambda \\ 2 & 4 \end{vmatrix} = 0$$

$$(4-\lambda)(\lambda^2 - 10\lambda + 21) - 2(\lambda - 3) + 3(2\lambda - 6) = 0$$

$$(4-\lambda)(\lambda-3)(\lambda-7) - 2(\lambda-3) + 6(\lambda-3) = 0$$

$$(\lambda-3)\left[(4-\lambda)(\lambda-7) - 2 + 6\right] = 0$$

$$-(\lambda-3)(\lambda^2 - 11\lambda + 24) = 0$$

$$-(\lambda-3)(\lambda-8)(\lambda-3) = 0$$

Eigenvalues: 3 and 8

3 - eigenspace $(A\vec{x} = \lambda\vec{x})$

$(A-\lambda I)\vec{x} = \vec{0}$ $\quad \lambda = 3$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ By observation $x_1 + 2x_2 + 3x_3 = 0$

Eigen vectors have form: $\begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\quad$ check ✓

8 - eigenspace

$$\begin{bmatrix} -4 & 2 & 3 \\ -1 & -7 & -3 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \left[\begin{array}{ccc|c} -4 & 2 & 3 & 0 \\ -1 & -7 & -3 & 0 \\ 2 & 4 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

eigenvectors: $x_3\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$ $\therefore$ Basis: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$.

$\therefore$ A couple of diagonalitations of A are PDP^{-1} where:

Option 1

$$P = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Option 2

$$P = \begin{bmatrix} -3 & 4 & -2 \\ 0 & -4 & 1 \\ 1 & 8 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Check $\quad P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 7 & 3 \\ -2 & -4 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

$$PDP^{-1} = \frac{1}{5}\begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix}\begin{bmatrix} 1 & 7 & 3 \\ -2 & -4 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -6 & -9 & 8 \\ 3 & 0 & -8 \\ 0 & 3 & 16 \end{bmatrix}\begin{bmatrix} 1 & 7 & 3 \\ -2 & -4 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 20 & 10 & 15 \\ -5 & 5 & -15 \\ 10 & 20 & 45 \end{bmatrix}$$

$= A$ ✓

Example

Diagonalize B where $B=\begin{bmatrix}3 & -1 & -2\\ 2 & 0 & -2\\ 2 & -1 & -1\end{bmatrix}$; use the result to simplify B^n where n is a natural number.

Characteristic equation: $\det(B-\lambda I)=0$

$$\begin{vmatrix}3-\lambda & -1 & -2\\ 2 & -\lambda & -2\\ 2 & -1 & -1-\lambda\end{vmatrix}=0;\quad \lambda=0 \text{ or } \lambda=1 \text{ (calc).}$$

0 – Eigenspace

$B\vec{x}=0\vec{x} \Rightarrow (B-0I)\vec{x}=\vec{0}$

$$\left[\begin{array}{ccc|c}3 & -1 & -2 & 0\\ 2 & 0 & -2 & 0\\ 2 & -1 & -1 & 0\end{array}\right]\sim\left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

General Solution
$$\begin{cases}x_1=x_3\\ x_2=x_3\\ x_3 \text{ is free}\end{cases}$$

Basis: $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

1 – Eigenspace

$(B-1I)\vec{x}=\vec{0}$

$$\left[\begin{array}{ccc|c}2 & -1 & -2 & 0\\ 2 & -1 & -2 & 0\\ 2 & -1 & -2 & 0\end{array}\right]$$

Gen. Ignore $2x_1-x_2-2x_3=0$

$x_1=\frac{1}{2}x_2+x_3$

Vectors in the space have form $x_2\begin{bmatrix}1/2\\1\\0\end{bmatrix}+x_3\begin{bmatrix}1\\0\\1\end{bmatrix}$

Basis: $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$

Diagonalization: $B=PDP^{-1}$ where

$$P=\begin{bmatrix}1 & 1 & 1\\ 2 & 0 & 1\\ 0 & 1 & 1\end{bmatrix};\ D=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix};\ P^{-1}=\begin{bmatrix}1 & 0 & -1\\ 2 & -1 & -1\\ -2 & 1 & 2\end{bmatrix}$$

$$\begin{aligned}B^2&=(PDP^{-1})(PDP^{-1})\\ &=PD(P^{-1}P)DP^{-1}\\ &=PDIDP^{-1}\\ &=PD^2P^{-1}\end{aligned}$$

$$\begin{aligned}B^3&=B^2B\\ &=(PD^2P^{-1})(PDP^{-1})\\ &=PD^2(P^{-1}P)DP^{-1}\\ &=PD^3P^{-1}\\ &\ \vdots\\ B^n&=PD^nP^{-1}\end{aligned}$$

$$\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}=\begin{bmatrix}a^2 & 0\\ 0 & b^2\end{bmatrix}\cdots\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}^n=\begin{bmatrix}a^n & 0\\ 0 & b^n\end{bmatrix}$$

$$B^n = P D^n P^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 0^n \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

HOLY COW!

$B^n = B \ \forall n$

B is called Idempotent.

C.P. $(\lambda-6)^4(\lambda-3)^2$

6-Eigenspace basis

exactly 4-vectors or exactly 3-vectors or 2 vectors or exactly 1-vector

3-Eigenspace

dimension is 2 or 1

Example

What happens when we try to diagonalize M where $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$?

Characteristic equation: $\det(M-\lambda I)=0$

$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix} = 0$; $\lambda = 1$ or $\lambda = 2$ (calculator)

twice once

From our calculator we can infer that

$\det(M-\lambda I) = (\lambda-1)^2(\lambda-2)$

So, if one of the eigenspaces has two vectors in its basis, its $\lambda = 1$.

1-eigenspace

$(M-1I)\vec{x} = \vec{0}$

$\begin{bmatrix} -1 & 1 & 0 & : & 0 \\ 0 & -1 & 1 & : & 0 \\ 2 & -5 & 3 & : & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & : & 0 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$

General sol. $\begin{cases} x_1 = x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \end{cases}$ Basis: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

Example

Find the eigenvalues for each matrix.

$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ -1 & 12 & 3 & 0 \\ 4 & 4 & 2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -1.9 & 14.4 & -8.4 & 34.8 \\ 1.6 & -2.7 & 3.2 & -1.6 \\ 1.2 & -8.0 & 4.7 & -18.2 \\ 1.6 & -1.6 & 3.2 & -2.7 \end{bmatrix}$

Since 1-eigenspace is 1-d, and 2-eigenspace is 1-d, we cannot come up with 3 linearly independent eigenvectors to create P. I cannot create an invertible P. M is not diagonalizable

Example

Diagonalize T where $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Characteristic equation: $\det(T - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \implies \lambda^2 + 1 = 0$$
$$\implies \lambda = \pm i$$

i - eigenspace

$$T\vec{x} = i\vec{x} \implies (T - \lambda I)\vec{x} = \vec{0}$$
$$\implies \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_1 = -i x_2$$

A Basis: $\left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} \right\}$

Check: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix}$
$= i \begin{bmatrix} 1 \\ i \end{bmatrix}$

$-i$ - eigenspace

$$(T - \lambda I)\vec{x} = \vec{0} \implies \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \quad x_1 = i x_2$$

A Basis: $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$

$\therefore$ A diagonalization of T is PDP^{-1} where $P = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$, $D = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$,

$P^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}$

$PDP^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
$= T$ ✓