Vector Spaces

A vector space is an algebraic structure consisting of a set of objects (V), called vectors, coupled with two operations called vector addition and scalar multiplication. In order to be considered a vector space, the set and operations must satisfy the ten defining axioms listed below.

The most commonly encountered vector spaces are the sets of all $n \times 1$ column vectors, $\mathbb{R}^n$, coupled with traditional vector addition and scalar multiplication; these type vector spaces are called ***Euclidean spaces***. However, several other sets and operations satisfy the axioms of a vector space, so to avoid repetitiveness mathematicians keep the definition of a vector space as general as possible so that the proof of additional properties covers all such structures.

Vector spaces are but one of many algebraic structures studied by mathematicians. Some other common algebraic structures are groups, rings, and fields. Each of these structures begins with a set of objects and one or more operation. Each of them requires closure, but the other defining properties vary from structure to structure.

Definition of a Vector Space

A ***vector space*** is a nonempty set of objects (called ***vectors***), V, over which we define two operations (***vector addition*** and ***scalar multiplication***) that satisfy each of the following properties

1. The set V is ***closed over vector addition***; i.e., $\vec{u} + \vec{v} \in V \;\; \forall \;\; \vec{u}, \vec{v} \in V$.
2. Vector addition is ***commutative***; i.e., $\vec{u} + \vec{v} = \vec{v} + \vec{u} \;\; \forall \;\; \vec{u}, \vec{v} \in V$.
3. Vector addition is ***associative***; i.e., $\left(\vec{u} + \vec{v}\right) + \vec{w} = \vec{u} + \left(\vec{v} + \vec{w}\right) \;\; \forall \;\; \vec{u}, \vec{v}, \vec{w} \in V$.
4. There is an ***additive identity*** (zero vector) in V; i.e., $\exists \, \vec{0} \in V \ni \vec{u} + \vec{0} = \vec{u} \;\; \forall \;\; \vec{u} \in V$.
5. Each vector in V has an ***additive inverse***; i.e., $\forall \, \vec{u} \in V, \exists -\vec{u} \in V \ni \vec{u} + \left(-\vec{u}\right) = \vec{0}$.
6. The set V is ***closed over scalar multiplication***; i.e., $c\,\vec{u} \in V \;\; \forall \;\; \vec{u} \in V$ and $c \in \mathbb{R}$.
7. Scalar multiplication distributes over vector addition; i.e.,
$$c\left(\vec{u} + \vec{v}\right) = c\,\vec{u} + c\,\vec{v} \;\; \forall \;\; \vec{u}, \vec{v} \in V \text{ and } c \in \mathbb{R}.$$
8. Scalar multiplication distributes over real number addition; i.e.,
$$\left(c + d\right)\vec{u} = c\,\vec{u} + d\,\vec{u} \;\; \forall \, \vec{u} \in V \text{ and } c, d \in \mathbb{R}.$$
9. Scalar and real number multiplications associate; i.e.,
$$\left(c\,d\right)\vec{u} = c\left(d\,\vec{u}\right) \forall \, \vec{u} \in V, \text{ and } c, d \in \mathbb{R}.$$
10. The number one acts as a ***scalar multiplicative identity***; i.e., $1\vec{u} = \vec{u} \;\; \forall \;\; \vec{u} \in V$.

Let's prove a couple of elementary properties of vector spaces.

Theorem: The additive identity of a given vector space is unique. Proof by contradiction:

Assume V has two additive identities, call them $\vec{0}_1$ & $\vec{0}_2$. Since $\vec{0}_1$ is an additive identity, $\vec{0}_2 + \vec{0}_1 = \vec{0}_2$ (axiom 4). Since $\vec{0}_2$ is an additive identity, $\vec{0}_1 + \vec{0}_2 = \vec{0}_1$ (axiom 4). But from Axiom 2, $\vec{0}_1 + \vec{0}_2 = \vec{0}_2 + \vec{0}_1$, ergo

$\vec{0}_1 = \vec{0}_2$. ✓

Theorem: $\forall \vec{u} \in V, -1\vec{u} = -\vec{u}$

$-1\vec{u}$ is scalar multiplication between -1 & $\vec{u}$

$-\vec{u}$ is the additive inverse of $\vec{u}$. Totally different meanings – the don't necessarily have to be equal. We need to show that $\vec{u} + (-1\vec{u}) = \vec{0}$.

Lemma: $0\vec{u} = \vec{0}\ \forall \vec{u}$
problem 27 in book

$$\begin{aligned} \vec{u} + (-1\vec{u}) &= 1\vec{u} + (-1\vec{u}) && \text{Axiom 10} \\ &= (1 + (-1))\vec{u} && \text{Axiom 8} \\ &= 0\vec{u} && \text{math} \\ &= \vec{0} && \text{lemma problem 27.} \end{aligned}$$

It is proven in higher level linear algebra classes that each finite-dimensional vector space is equivalent in structure to some R^n; i.e., the elements of V can be paired up with the elements of some R^n in such a way that addition and scalar multiplication are preserved. This pairing is represented by a one-to-one/onto linear transformation $T: V \to \mathbb{R}^n$. In broader terms, V & $\mathbb{R}^n$ are called isomorphic. Isomorphic structions always have bijective mappings that preserve the operations.

Example

Let V be the set of 2 x 2 matrices with vector addition defined as matrix addition and scalar multiplication defined in the traditional way. Show that this vector set is structurally equivalent (isomorphic) to some R^n.

$$T: V \to \mathbb{R}^4 \text{ where } T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

It's "obvious" that this mapping is bijective and maintains the same addition & scalar multiplication outcomes.

Example

Let P_2 be the set of all polynomials of degree two or less with vector addition defined as polynomial addition and scalar multiplication defined by distribution of the scalar over each term. Show that this vector set is $\underbrace{\text{structurally equivalent to}}_{\text{isomorphic}}$ some R^n.

$T: P_2 \to \mathbb{R}^3$ where $T(ax^2+bx+c) = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$.

Obviously bijective. As for preservation of operations...

$(a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2) = (a_1+a_2)x^2 + (b_1+b_2)x + (c_1+c_2)$

$\begin{bmatrix} c_1 \\ b_1 \\ a_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ b_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} c_1+c_2 \\ b_1+b_2 \\ a_1+a_2 \end{bmatrix}$ likewise for scalar multiplication

Definition of a Subspace

A ***subspace*** of the vector space V is a nonempty subset of V that is closed over vector addition and scalar multiplication.

Theorem: The additive identity of V is in every subspace of V.

By definition, a subspace of V is nonempty. Let $\vec{v}$ be any element in the subspace, then $0\vec{v}$ is also in the subspace (closure over scalar multiplication) and by lemma 1, $0\vec{v} = \vec{0}$.

QED

Example

Describe, using words, the membership qualification for each of the following sets. Then decide whether or not the given set constitutes a subspace of $\mathbb{R}^3$. In each case, prove your contention. For each set a and b can represent any real number.

$$W = \left\{ \begin{bmatrix} a \\ b \\ 3 \end{bmatrix} \right\}$$

In order for $\vec{v} \in W$, third entry must be 3.

$\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \in W$, $\begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix} \in W$, $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix}$.

$\begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix} \notin W$. W is not closed over vector addition and as such is not a subspace of $\mathbb{R}^3$.

$$X = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \right\}$$

Vectors in X have third entries that are zero.

Let $\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \in X$ & $\begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} \in X$. Then:

$$\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ 0 \end{bmatrix} \in X$$

and

$$k \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \\ 0 \end{bmatrix} \in X$$

$\therefore$ X is a subspace of $\mathbb{R}^3$. In fact, it's really $\mathbb{R}^2$.

Preliminary exploration: $\begin{bmatrix} 2 \\ 3 \\ 17 \end{bmatrix} + \begin{bmatrix} -4 \\ 6 \\ 26 \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ 43 \end{bmatrix}$ in the set.

$$Y = \left\{ \begin{bmatrix} a \\ b \\ a+5b \end{bmatrix} \right\}$$

Vectors in Y have third entries that are the first entry plus five times the second entry.

Let $\begin{bmatrix} x_1 \\ x_2 \\ x_1+5x_2 \end{bmatrix} \in Y$ and $\begin{bmatrix} y_1 \\ y_2 \\ y_1+5y_2 \end{bmatrix} \in Y$, then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_1+5x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_1+5y_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ (x_1+5x_2)+(y_1+5y_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ (x_1+y_1)+(5x_2+5y_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ (x_1+y_1)+5(x_2+y_2) \end{bmatrix} \in Y$$

$$k\begin{bmatrix} x_1 \\ x_2 \\ x_1+5x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \\ k(x_1+5x_2) \end{bmatrix}$$

$$= \begin{bmatrix} kx_1 \\ kx_2 \\ kx_1 + k\cdot 5x_2 \end{bmatrix}$$

$$= \begin{bmatrix} kx_1 \\ kx_2 \\ kx_1 + 5\cdot(kx_2) \end{bmatrix} \in Y$$

$\therefore$ Y is a subspace of $\mathbb{R}^3$.

$$Z = \left\{ \begin{bmatrix} a \\ b \\ a+5 \end{bmatrix} \right\}$$

Vectors in Z have third entries that are five more than their first entries.

$\begin{bmatrix} 2 \\ 7 \\ 7 \end{bmatrix} \in Z$ and $\begin{bmatrix} -8 \\ 4 \\ -3 \end{bmatrix} \in Z$

$$\begin{bmatrix} 2 \\ 7 \\ 7 \end{bmatrix} + \begin{bmatrix} -8 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -6 \\ 11 \\ 4 \end{bmatrix} \notin Z$$

$\therefore$ Z is <u>not</u> closed over scalar addition and, consequently, is <u>not</u> a subspace of $\mathbb{R}^3$.

Theorem

The span of a non-empty set of vectors from V is a subspace of V.

Let's prove the theorem in the box.

Let $W = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \subset V$. Let $\vec{x} \in \operatorname{span}(W)$ and $\vec{y} \in \operatorname{span}(W)$.

Then $\exists$ scalars $x_1, \ldots, x_p$ and $y_1, \ldots, y_p$ such that

$$\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p \quad \& \quad \vec{y} = y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_p\vec{v}_p.$$

Then:
$$\vec{x} + \vec{y} = (x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p) + (y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_p\vec{v}_p)$$
$$= (x_1\vec{v}_1 + y_1\vec{v}_1) + (x_2\vec{v}_2 + y_2\vec{v}_2) + \cdots + (x_p\vec{v}_p + y_p\vec{v}_p)$$
$$= (x_1 + y_1)\vec{v}_1 + (x_2 + y_2)\vec{v}_2 + \cdots + (x_p + y_p)\vec{v}_p \in \operatorname{span}(W)$$

and
$$k\vec{x} = k(x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p)$$
$$= k \cdot x_1\vec{v}_1 + k \cdot x_2\vec{v}_2 + \cdots + k \cdot x_p\vec{v}_p$$
$$= (kx_1) \cdot \vec{v}_1 + (kx_2) \cdot \vec{v}_2 + \cdots + (kx_p)\vec{v}_p \in \operatorname{span}(W)$$

QED

Example

Find a spanning set for the subspace of $\mathbb{R}^3$ consisting of vectors of form $\left\{\begin{bmatrix} a \\ b \\ a+5b \end{bmatrix}\right\} = Y$.

We established that this is a subspace on page 5.

$$\begin{bmatrix} a \\ b \\ a+5b \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 5b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$$

This established that Y is the span of $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}\right\}$.
So by the theorem up top, we have again established that Y is a subspace of $\mathbb{R}^3$.

Bonus: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$ obviously form a linearly independent set, so $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}\right\}$ is called a basis for Y. Since the basis has two vectors, Y is isomorphic to $\mathbb{R}^2$.

Example

Show that solutions to the equation $\begin{bmatrix} 3 & -7 \\ -6 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\overbrace{\text{form a subspace of } \mathbb{R}^2}^{a}$ and then $\overbrace{\text{find a set}}^{b}$ of vectors that span the subspace.

a) Let $A = \begin{bmatrix} 3 & -7 \\ -6 & 14 \end{bmatrix}$. Solutions to $A\vec{x} = \vec{0}$ are called the <u>null space</u> of A. Let $\vec{x} \in nul(A)$ and $\vec{y} \in nul(A)$. Then

$$A\cdot(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$$

So $\vec{x}+\vec{y} \in nul(A)$

$$A\cdot(k\vec{x}) = k\cdot(A\vec{x}) = k\vec{0} = \vec{0}$$

So $k\vec{x} \in nul(A)$

QED

b) $A\vec{x} = \vec{0} \Rightarrow \begin{cases} 3x_1 - 7x_2 = 0 \\ -6x_1 + 14x_2 = 0 \end{cases}$

$$\left[\begin{array}{cc|c} 3 & -7 & 0 \\ -6 & 14 & 0 \end{array}\right] \xrightarrow{2R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 3 & -7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution to $A\vec{x} = \vec{0}$ is $\begin{cases} x_1 = \frac{7}{3}x_2 \\ x_2 \text{ is free} \end{cases}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 7/3 \\ 1 \end{bmatrix}$$

$$\therefore nul(A) = span\left(\left\{\begin{bmatrix} 7/3 \\ 1 \end{bmatrix}\right\}\right)$$

The null space of A is <u>one</u> - <u>dimensional</u> and as such is Isomorphic to $\mathbb{R}^1$.

Definition of (and a theorem about) Column Space

The ***column space*** of a matrix A is the set of all linear combinations of the columns of A. If A is an $m \times n$ matrix, then $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$.

Example

Let $A=\begin{bmatrix} 3 & 4 & -2 \\ -1 & 6 & 4 \\ 5 & 14 & 0 \end{bmatrix}$. Determine whether or not $\mathrm{Col}(A)$ is all of $\mathbb{R}^3$.

Is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathrm{col}(A)$ $\forall \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$

i.e. does $x_1\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 6 \\ 14 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ have a solution for all $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$?

i.e. do the column vectors of A span $\mathbb{R}^3$?

This is property h of Theorem 8, so we can establish the answer using property m.

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & 6 & 4 \\ 5 & 14 & 0 \end{vmatrix} = 0 \quad \therefore$$

The columns do <u>not</u> span $\mathbb{R}^3$.

Hence $\mathrm{col}(A) \neq \mathbb{R}^3$.

i.e. There is at least one vector from $\mathbb{R}^3$ that is not in the column space of A.

Example

Following up on the last example, determine whether or not $[5,5,10]^{\mathrm{T}} \in \mathrm{col}(A)$.

Can I write $\begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix}$ as a linear combination of the columns of A?

Does $x_1\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 6 \\ 14 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix}$ have at least one solution?

$$\left[\begin{array}{ccc|c} 3 & 4 & -2 & 5 \\ -1 & 6 & 4 & 5 \\ 5 & 14 & 0 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -14/11 & 0 \\ 0 & 1 & 5/11 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \leftarrow \text{contradiction}$$

$$\therefore \begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix} \notin \mathrm{col}(A)$$

Definition of (and a theorem about) Null Space

The ***null space*** of a matrix A is the set of all solutions to the equation $A\vec{x}=\vec{0}$. If A is an $m\times n$ matrix, then $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

Example

Let $A=\begin{bmatrix}2 & -2 & 3\\ 4 & 1 & 6\end{bmatrix}$. Determine whether or not $\begin{bmatrix}2 & 1 & 0\end{bmatrix}^{\mathrm{T}}$ is an element of either the column space or the null space of A. Also, find a spanning set for $\mathrm{nul}(A)$

Let $\vec{x}=\begin{bmatrix}2\\1\\0\end{bmatrix}$. $\vec{x}\notin \mathrm{col}(A)$ because $\mathrm{col}(A)\subset \mathbb{R}^2$ & $\vec{x}\in\mathbb{R}^3$.

is $\vec{x}\in \mathrm{nul}(A)$? Does $A\vec{x}=\vec{0}$?

$$A\vec{x}=\begin{bmatrix}2 & -2 & 3\\ 4 & 1 & 6\end{bmatrix}\begin{bmatrix}2\\1\\0\end{bmatrix}$$
$$=\begin{bmatrix}4-2+0\\ 8+1+0\end{bmatrix}$$
$$=\begin{bmatrix}2\\9\end{bmatrix}$$

$\therefore\ \vec{x}\notin \mathrm{nul}(A)$

explore the null space

$\begin{bmatrix}2 & -2 & 3\\ 4 & 1 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ is equivalent to the system $\begin{cases}2x_1-2x_2+3x_3=0\\ 4x_1+x_2+6x_3=0\end{cases}$

$$\left[\begin{array}{ccc|c}2 & -2 & 3 & 0\\ 4 & 1 & 6 & 0\end{array}\right]\sim\left[\begin{array}{ccc|c}1 & 0 & 3/2 & 0\\ 0 & 1 & 0 & 0\end{array}\right]$$

The general solution is $\begin{cases}x_1=-\frac{3}{2}x_3\\ x_2=0\\ x_3 \text{ is free}\end{cases}$

$\therefore$ vectors in the null space can be written

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_3\begin{bmatrix}-3/2\\0\\1\end{bmatrix}$$

$\therefore\ \left\{\begin{bmatrix}-3/2\\0\\1\end{bmatrix}\right\}$ spans $\mathrm{nul}(A)$.