Vector Spaces

Extra HW from section 2.8

A vector space is an algebraic structure consisting of a set of objects (V), called vectors, coupled with two operations called vector addition and scalar multiplication. In order to be considered a vector space, the set and operations must satisfy the ten defining axioms listed below.

The most commonly encountered vector spaces are the sets of all $n \times 1$ column vectors, $\mathbb{R}^n$, coupled with traditional vector addition and scalar multiplication; these type vector spaces are called ***Euclidean spaces***. However, several other sets and operations satisfy the axioms of a vector space, so to avoid repetitiveness mathematicians keep the definition of a vector space as general as possible so that the proof of additional properties covers all such structures.

Vector spaces are but one of many algebraic structures studied by mathematicians. Some other common algebraic structures are groups, rings, and fields. Each of these structures begins with a set of objects and one or more operation. Each of them requires closure, but the other defining properties vary from structure to structure.

Definition of a Vector Space

A ***vector space*** is a nonempty set of objects (called ***vectors***), V, over which we define two operations (***vector addition*** and ***scalar multiplication***) that satisfy each of the following properties

1. The set V is ***closed over vector addition***; i.e., $\vec{u} + \vec{v} \in V \ \forall \ \vec{u}, \vec{v} \in V$. (closure property)
2. Vector addition is ***commutative***; i.e., $\vec{u} + \vec{v} = \vec{v} + \vec{u} \ \forall \ \vec{u}, \vec{v} \in V$.
3. Vector addition is ***associative***; i.e., $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \ \forall \ \vec{u}, \vec{v}, \vec{w} \in V$.
4. There is an ***additive identity*** (zero vector) in V; i.e., $\exists\, \vec{0} \in V \ni \vec{u} + \vec{0} = \vec{u} \ \forall \ \vec{u} \in V$.
5. Each vector in V has an ***additive inverse***; i.e., $\forall\, \vec{u} \in V, \exists -\vec{u} \in V \ni \vec{u} + (-\vec{u}) = \vec{0}$.
6. The set V is ***closed over scalar multiplication***; i.e., $c\vec{u} \in V \ \forall \ \vec{u} \in V$ and $c \in \mathbb{R}$. (closure property)
7. Scalar multiplication distributes over vector addition; i.e.,
   $$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v} \ \forall \ \vec{u}, \vec{v} \in V \text{ and } c \in \mathbb{R}.$$
8. Scalar multiplication distributes over real number addition; i.e.,
   $$(c + d)\vec{u} = c\vec{u} + d\vec{u} \ \forall\, \vec{u} \in V \text{ and } c, d \in \mathbb{R}.$$
9. Scalar and real number multiplications associate; i.e.,
   $$(cd)\vec{u} = c(d\vec{u}) \ \forall\, \vec{u} \in V, \text{ and } c, d \in \mathbb{R}.$$
10. The number one acts as a ***scalar multiplicative identity***; i.e., $1\vec{u} = \vec{u} \ \forall \ \vec{u} \in V$.

Let's prove a couple of elementary properties of vector spaces.

Theorem: The additive identity of a given vector space is unique.

Suppose $\vec{0}_1$ and $\vec{0}_2$ are both additive identities of a given vector space. Since $\vec{0}_1$ is an additive identity, $\vec{0}_2 + \vec{0}_1 = \vec{0}_2$ and since $\vec{0}_2$ is an additive identity, $\vec{0}_1 + \vec{0}_2 = \vec{0}_1$.

But vector addition is commutative, so $\vec{0}_2 + \vec{0}_1 = \vec{0}_1 + \vec{0}_2$.

Ergo $\vec{0}_2 = \vec{0}_1$

Theorem: $\forall \vec{u} \in V, -1\vec{u} = -\vec{u}$

$-1\vec{u}$ is a scalar product

$-\vec{u}$ is the additive inverse of $\vec{u}$

It does not go without saying that these things are equal.

$\boxed{\text{what does } -\vec{u} \text{ mean actionably: } \vec{u} + (-\vec{u}) = \vec{0}}$

So we need to show that $\vec{u} + (-1 \cdot \vec{u}) = \vec{0}$.

$$\begin{aligned}\vec{u} + (-1 \cdot \vec{u}) &= 1 \cdot \vec{u} + (-1) \cdot \vec{u} && \text{(property 10)}\\ &= (1 + (-1)) \cdot \vec{u} && \text{(property 8)}\\ &= 0 \cdot \vec{u} = \vec{0} && \text{(lemma 1)} \leftarrow \text{Page 3}\end{aligned}$$

It is proven in higher level linear algebra classes that each finite-dimensional vector space is equivalent in structure to some R^n; i.e., the elements of V can be paired up with the elements of some R^n in such a way that addition and scalar multiplication are preserved. This pairing is represented by a one-to-one/onto linear transformation $T: V \to \mathbb{R}^n$.

Example

Let V be the set of 2 x 2 matrices with vector addition defined as matrix addition and scalar multiplication defined in the traditional way. Show that this vector set is structurally equivalent to some R^n.

V: elements are $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$

V is equivalent to $\mathbb{R}^4$.

$T: V \to \mathbb{R}^4$ under the rule

$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ T is obviously one-to-one and onto. QED.

We say that P_2 and $\mathbb{R}^3$ are isomorphic and T is called an isomorphism.

Example

Let P_2 be the set of all polynomials of degree two or less with vector addition defined as polynomial addition and scalar multiplication defined by distribution of the scalar over each term. Show that this vector set is structurally equivalent to some R^n.

$P_2 = \{ at^2 + bt + c \mid a, b, c \in \mathbb{R} \}$

P_2 is equivalent to $\mathbb{R}^3$

$T : P_2 \to \mathbb{R}^3$ via $T(at^2 + bt + c) = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$

T is obviously one-to-one and onto.

QED

Definition of a Subspace

A ***subspace*** of the vector space V is a nonempty subset of V that is closed over vector addition and scalar multiplication.

Theorem: The additive identity of V is in every subspace of V.

Lemma 1: $0 \cdot \vec{v} = \vec{0}$

From vector space property 4, we need to show that $\vec{u} + 0\vec{v} = \vec{u}\ \forall \vec{u}$.

(problem 27 in section 4.1)

Prove that $\vec{0} \in V,\ \forall V$.

Since V is a nonempty set, $\exists\, \vec{v} \in V$. Since V is closed over scalar multiplication,

$0 \cdot \vec{v} \in V$.

QED (lemma 1)

Example

Describe, using words, the membership qualification for each of the following sets. Then decide whether or not the given set constitutes a subspace of $\mathbb{R}^3$. In each case, prove your contention. For each set a and b can represent any real number.

$$W = \left\{ \begin{bmatrix} a \\ b \\ 3 \end{bmatrix} \right\}$$

The third entry (x_3) of every vector in this set is 3. This set is obviously not closed over vector addition.

$$\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \\ \text{not } 3 \end{bmatrix}$$

Likewise for scalar multiplication.

$$2\begin{bmatrix} 6 \\ 14 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 28 \\ \text{not } 3 \end{bmatrix}$$

$$X = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \right\}$$

X is closed over both operations. To be in X, x_3 must equal zero.

$$\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ 0 \end{bmatrix} \leftarrow$$ The resultant vector under addition is in the club

$$k\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ 0 \end{bmatrix} \leftarrow$$ The resultant vector under scalar multiplication is in the set.

X forms a proper subspace of $\mathbb{R}^3$

not all of $\mathbb{R}^3$.

Explore $\begin{bmatrix} 1 \\ 2 \\ 11 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$ looks like the set is closed.

$$Y = \left\{ \begin{bmatrix} a \\ b \\ a+5b \end{bmatrix} \right\}$$

The third entry of vectors in Y is the sum of the first entry and five times the second entry.

$$\begin{bmatrix} a_1 \\ b_1 \\ a_1+5b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ a_2+5b_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ (a_1+5b_1)+(a_2+5b_2) \end{bmatrix}$$

$$= \begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ (a_1+a_2)+5(b_1+b_2) \end{bmatrix} \in Y$$

Y does form a subspace of $\mathbb{R}^3$

proper

$$k \begin{bmatrix} a \\ b \\ a+5b \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ k(a+5b) \end{bmatrix}$$

$$= \begin{bmatrix} ka \\ kb \\ (ka)+5\cdot(kb) \end{bmatrix} \in Y$$

$$Z = \left\{ \begin{bmatrix} a \\ b \\ a+5 \end{bmatrix} \right\}$$

The third entry of vectors in Z is the first entry plus five.

Preliminary exploration: $\begin{bmatrix} 6 \\ 7 \\ 11 \end{bmatrix} + \begin{bmatrix} -14 \\ -2 \\ -9 \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix} \notin Z$

in Z   in Z

Z is not closed over vector addition.

Z is not a subspace of $\mathbb{R}^3$.

> **Theorem**
>
> The span of a set of vectors from V is a subspace of V.

Let's prove the theorem in the box.

Example

Find a spanning set for the subspace of $\mathbb{R}^3$ consisting of vectors of form $\left\{\begin{bmatrix} a \\ b \\ a+5b \end{bmatrix}\right\}$. This is set Z

Vectors in Z can be written thus:

$$\begin{bmatrix} a \\ b \\ a+5b \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 5b \end{bmatrix}$$

$$= a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$$

Ergo $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}\right\}$ spans Z.

Since this set is (obviously) linearly independent

it further forms a basis for Z. Z is two-dimensional.

Z is isomorphic to $\mathbb{R}^2$.

Example (Part 1) (Part 2)

Show that solutions to the equation $\begin{bmatrix} 3 & -7 \\ -6 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ form a subspace of $\mathbb{R}^2$ and then find a set of vectors that span the subspace.

Part 2

What we are doing is finding a basis for the null space of A where $A = \begin{bmatrix} 3 & -7 \\ -6 & 14 \end{bmatrix}$.

$\left[\begin{array}{cc|c} 3 & -7 & 0 \\ -6 & 14 & 0 \end{array}\right] \xrightarrow{2R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 3 & -7 & 0 \\ 0 & 0 & 0 \end{array}\right]$ clearly $x_1 = \frac{7}{3}x_2$

So the null space is $\left\{ \begin{bmatrix} \frac{7}{3}x_2 \\ x_2 \end{bmatrix} \middle| \, x_2 \in \mathbb{R} \right\}$. A basis for null(A) is $\left\{ \begin{bmatrix} 7/3 \\ 1 \end{bmatrix} \right\}$ a different basis is $\left\{ \begin{bmatrix} 7 \\ 3 \end{bmatrix} \right\}$.

Definition of (and a theorem about) Column Space

The ***column space*** of a matrix A is the set of all linear combinations of the columns of A. If A is an $m \times n$ matrix, then $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$.

Example

Let $A = \begin{bmatrix} 3 & 4 & -2 \\ -1 & 6 & 4 \\ 5 & 14 & 0 \end{bmatrix}$. Determine whether or not $\text{Col}(A)$ is all of $\mathbb{R}^3$.

We need to determine if $A\vec{x} = \vec{b}$ always has at least one solution $\forall\ \vec{b} \in \mathbb{R}^3$. This is Akin to Saying $T(\vec{x}) = A\vec{x}$ is onto $\mathbb{R}^3$. That is property "i" of Theorem Awesome which is equivalent to property m. Ergo, col(A) is all of $\mathbb{R}^3$ iff $\det(A) \neq 0$.

$$\det(A) = \begin{vmatrix} 3 & 4 & -2 \\ -1 & 6 & 4 \\ 5 & 14 & 0 \end{vmatrix} = -2(-1)^4 \begin{vmatrix} -1 & 6 \\ 5 & 14 \end{vmatrix} + 4(-1)^5 \begin{vmatrix} 3 & 4 \\ 5 & 14 \end{vmatrix} + 0(-1)^6 \begin{vmatrix} 3 & 4 \\ -1 & 6 \end{vmatrix}$$

$$= -2(-44) - 4(42-20) = 0 \therefore$$

$\text{col}(A) \neq \mathbb{R}^3$

Example

Following up on the last example, determine whether or not $[5,5,10]^T \in \text{col}(A)$.

We need to try to solve

$$x_1 \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 6 \\ 14 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 4 & -2 & 5 \\ -1 & 6 & 4 & 5 \\ 5 & 14 & 0 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -14/11 & 0 \\ 0 & 1 & 5/11 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \Leftarrow \text{contradiction } 0 \neq 1$$

$$\therefore \begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix} \notin \text{col}(A)$$

Definition of (and a theorem about) Null Space

The ***null space*** of a matrix A is the set of all solutions to the equation $A\vec{x}=\vec{0}$. If A is an $m\times n$ matrix, then $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

Example

Let $A=\begin{bmatrix}2 & -2 & 3\\ 4 & 1 & 6\end{bmatrix}$. Determine whether or not $\begin{bmatrix}2 & 1 & 0\end{bmatrix}^T$ is an element of either the column space or the null space of A. Also, find a spanning set for $\mathrm{nul}(A)$

Subset

$\begin{bmatrix}2\\1\\0\end{bmatrix}$ is obviously not in column space ($\mathrm{col}(A)\subset\mathbb{R}^2$)

$\begin{bmatrix}2 & -2 & 3\\ 4 & 1 & 6\end{bmatrix}\begin{bmatrix}2\\1\\0\end{bmatrix}=\begin{bmatrix}2\\9\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}$, so $\begin{bmatrix}2\\1\\0\end{bmatrix}\notin \mathrm{null}(A)$

$$\left[\begin{array}{ccc|c}2 & -2 & 3 & 0\\ 4 & 1 & 6 & 0\end{array}\right] \xrightarrow{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c}2 & -2 & 3 & 0\\ 0 & 5 & 0 & 0\end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_2\to R_2} \left[\begin{array}{ccc|c}2 & -2 & 3 & 0\\ 0 & 1 & 0 & 0\end{array}\right]$$

$$\xrightarrow{2R_2+R_1\to R_1} \left[\begin{array}{ccc|c}2 & 0 & 3 & 0\\ 0 & 1 & 0 & 0\end{array}\right]$$

General Solution: $\begin{cases} x_1 = -3/2\, x_3\\ x_2 = 0\\ x_3 \text{ is free}\end{cases}$

Since the solution has one free variable, the null space of A has bases with one variable.

$$\begin{bmatrix}-3/2\, x_3\\ 0\\ x_3\end{bmatrix} = x_3\begin{bmatrix}-3/2\\ 0\\ 1\end{bmatrix}$$

Any non-zero vector in this null space forms a basis for the entire null space.

To wit $\left\{\begin{bmatrix}-9\\0\\6\end{bmatrix}\right\}$ forms a basis for $\mathrm{null}(A)$.