Vector Spaces

A vector space is an algebraic structure consisting of a set of objects (V), called vectors, coupled with two operations called vector addition and scalar multiplication. In order to be considered a vector space, the set and operations must satisfy the ten defining axioms listed below.

The most commonly encountered vector spaces are the sets of all $n \times 1$ column vectors, $\mathbb{R}^n$, coupled with traditional vector addition and scalar multiplication; these type vector spaces are called ***Euclidean spaces***. However, several other sets and operations satisfy the axioms of a vector space, so to avoid repetitiveness mathematicians keep the definition of a vector space as general as possible so that the proof of additional properties covers all such structures.

Vector spaces are but one of many algebraic structures studied by mathematicians. Some other common algebraic structures are groups, rings, and fields. Each of these structures begins with a set of objects and one or more operation. Each of them requires closure, but the other defining properties vary from structure to structure.

Definition of a Vector Space

A ***vector space*** is a nonempty set of objects (called ***vectors***), V, over which we define two operations (***vector addition*** and ***scalar multiplication***) that satisfy each of the following properties

1. The set V is ***closed over vector addition***; i.e., $\vec{u} + \vec{v} \in V \;\; \forall \;\; \vec{u}, \vec{v} \in V$.
2. Vector addition is ***commutative***; i.e., $\vec{u} + \vec{v} = \vec{v} + \vec{u} \;\; \forall \;\; \vec{u}, \vec{v} \in V$.
3. Vector addition is ***associative***; i.e., $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \;\; \forall \;\; \vec{u}, \vec{v}, \vec{w} \in V$.
4. There is an ***additive identity*** (zero vector) in V; i.e., $\exists\, \vec{0} \in V \ni \vec{u} + \vec{0} = \vec{u} \;\; \forall \;\; \vec{u} \in V$.
5. Each vector in V has an ***additive inverse***; i.e., $\forall\, \vec{u} \in V, \exists -\vec{u} \in V \ni \vec{u} + (-\vec{u}) = \vec{0}$.
6. The set V is ***closed over scalar multiplication***; i.e., $c\,\vec{u} \in V \;\; \forall \;\; \vec{u} \in V$ and $c \in \mathbb{R}$.
7. Scalar multiplication distributes over vector addition; i.e.,

   $$c(\vec{u} + \vec{v}) = c\,\vec{u} + c\,\vec{v} \;\; \forall \;\; \vec{u}, \vec{v} \in V \text{ and } c \in \mathbb{R}.$$
8. Scalar multiplication distributes over real number addition; i.e.,

   $$(c + d)\vec{u} = c\,\vec{u} + d\,\vec{u} \;\; \forall\, \vec{u} \in V \text{ and } c, d \in \mathbb{R}.$$
9. Scalar and real number multiplications associate; i.e.,

   $$(c\,d)\vec{u} = c(d\,\vec{u})\, \forall\, \vec{u} \in V, \text{ and } c, d \in \mathbb{R}.$$
10. The number one acts as a ***scalar multiplicative identity***; i.e., $1\vec{u} = \vec{u} \;\; \forall \;\; \vec{u} \in V$.

Let's prove a couple of elementary properties of vector spaces.

Theorem: The additive identity of a given vector space is unique.

Let's assume that $\vec{v}$ and $\vec{w}$ are distinct additive identities. Then, treating $\vec{v}$ as $\vec{0}$

$\vec{w} + \vec{v} = \vec{w}$

treating $\vec{w}$ as $\vec{0}$

$\vec{v} + \vec{w} = \vec{v}$

But... $\vec{w} + \vec{v} = \vec{v} + \vec{w}$ (property 2), so this implies that $\vec{w} = \vec{v}$

Theorem: $\forall\, \vec{u} \in V, -1\vec{u} = -\vec{u}$ $-\vec{u}$ is the symbol for that additive inverse of $\vec{u}$

We need to show: $\vec{u} + (-1\vec{u}) = \vec{0}$.

$$
\begin{aligned}
\vec{u} + (-1\vec{u}) &= 1\vec{u} + (-1\vec{u}) \\
&= (1 + (-1))\vec{u} \quad \text{(property 8)} \\
&= 0\vec{u} \\
&= \vec{0} \quad \text{(problem in book)}
\end{aligned}
$$

It is proven in higher level linear algebra classes that each finite-dimensional vector space is equivalent in structure to some R^n; i.e., the elements of V can be paired up with the elements of some R^n in such a way that addition and scalar multiplication are preserved. This pairing is represented by a one-to-one/onto linear transformation $T : V \to \mathbb{R}^n$. V and $\mathbb{R}^n$ are called isomorphic.

Example

Let V be the set of 2×2 matrices with vector addition defined as matrix addition and scalar multiplication defined in the traditional way. Show that this vector set is structurally equivalent to some R^n.

We need to find $T : M_{2\times 2} \to \mathbb{R}^4$ where T is one-to-one and onto.

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$ This transformation is obviously one-to-one and onto $\mathbb{R}^4$!

Example

Let P_2 be the set of all polynomials of degree two or less with vector addition defined as polynomial addition and scalar multiplication defined by distribution of the scalar over each term. Show that this vector set is structurally equivalent to some R^n.

vectors: $\{a + bt + ct^2\}$

$T(a + bt + ct^2) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

This linear transformation is obviously one-to-one and onto $\mathbb{R}^3$

> **Definition of a Subspace**
>
> A ***subspace*** of the vector space V is a nonempty subset of V that is closed over vector addition and scalar multiplication.

Theorem: The additive identity of V is in every subspace of V.

Call the subspace H. H has at least one vector in it, call that vector $\vec{u}$.

Because H is closed over scalar multiplication,

$0\vec{u} \in H$.

QED

(because $0\vec{u} = \vec{0}$)

Example

Decide whether or not each of the following sets constitute subspaces of $\mathbb{R}^3$. In each case, prove your contention. For each set a and b can represent any real number.

$\left\{ \begin{bmatrix} a \\ b \\ 3 \end{bmatrix} \right\}$

This set does not form a subspace; it's not closed over vector addition (or scalar multiplication).

$\begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 6 \end{bmatrix}$ QED

$\left\{\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}\right\}$

This does form a subspace. Call the set H.

$$\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ 0 \end{bmatrix} \in H$$

$$c\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} = \begin{bmatrix} ca_1 \\ cb_1 \\ 0 \end{bmatrix} \in H$$

QED

$\left\{\begin{bmatrix} a \\ b \\ a+5b \end{bmatrix}\right\}$

Let's call the set H.

$$\begin{bmatrix} a_1 \\ b_1 \\ a_1 + 5b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ a_2 + 5b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ (a_1 + a_2) + 5(b_1 + b_2) \end{bmatrix} \in H$$

$$c\begin{bmatrix} a_1 \\ b_1 \\ a_1 + 5b_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ cb_1 \\ c(a_1 + 5b_1) \end{bmatrix} = \begin{bmatrix} ca_1 \\ cb_1 \\ ca_1 + 5(cb_1) \end{bmatrix} \in H$$

QED

$\left\{\begin{bmatrix} a \\ b \\ a+5 \end{bmatrix}\right\}$

Call the set H.

$$\underbrace{\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}}_{\text{in } H} + \underbrace{\begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}}_{\text{in } H} = \begin{bmatrix} -2 \\ 6 \\ 8 \end{bmatrix} \notin H$$

$8 \neq -2 + 5$

Theorem

The span of a set of vectors from V is a subspace of V.

Let's prove the theorem in the box. Consider $\text{span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$

$$(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) + (d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n)$$
$$= (c_1+d_1)\vec{v}_1 + (c_2+d_2)\vec{v}_2 + \cdots + (c_n+d_n)\vec{v}_n \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$$

(property 8)

$$k(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = (kc_1)\vec{v}_1 + \cdots + (kc_n)\vec{v}_n \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$$

(properties 7 and 9)

QED

Example

Find a spanning set for the subspace of $\mathbb{R}^3$ consisting of vectors of form $\left\{\begin{bmatrix} a \\ b \\ a+5b \end{bmatrix}\right\}$.

$$\begin{bmatrix} a \\ b \\ a+5b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$$

$\therefore$ The subspace is $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}\right\}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} ax_1 + bx_2 = 0 \\ cx_1 + dx_2 = 0 \end{cases}$$

Example

Show that solutions to the equation $\begin{bmatrix} 3 & -7 \\ -6 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ form a subspace of $\mathbb{R}^2$ and then find a set of vectors that span the subspace.

Let's define $A = \begin{bmatrix} 3 & -7 \\ -6 & 14 \end{bmatrix}$. The solutions to $A\vec{x} = \vec{0}$ are called the <u>null space</u> of A.

Assume $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \text{nul}(A)$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \text{nul}(A)$.

Then:

$$\begin{bmatrix} 3 & -7 \\ -6 & 14 \end{bmatrix}\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} = \begin{bmatrix} 3(x_1+y_1) - 7(x_2+y_2) \\ -6(x_1+y_1) + 14(x_2+y_2) \end{bmatrix}$$

Similarly, $k\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \text{nul}(A)$

$$= \begin{bmatrix} 3x_1 - 7x_2 \\ -6x_1 + 14x_2 \end{bmatrix} + \begin{bmatrix} 3y_1 - 7y_2 \\ -6y_1 + 14y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -7 \\ -6 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 & -7 \\ -6 & 14 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & -7 & 0 \\ -6 & 14 & 0 \end{array}\right] \xrightarrow{2R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 3 & -7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

vector in the null space satisfy $3x_1 - 7x_2 = 0$, so $x_1 = \frac{7}{3}x_2$

Ergo, $\text{nul}(A) = \text{Span}\left\{\begin{bmatrix} 7 \\ 3 \end{bmatrix}\right\}$

> **Definition of (and a theorem about) Column Space**
>
> The ***column space*** of a matrix A is the set of all linear combinations of the columns of A. If A is an $m \times n$ matrix, then $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$.

Example

Let $A=\begin{bmatrix}3 & 4 & -2\\ -1 & 6 & 4\\ 5 & 14 & 0\end{bmatrix}$. Determine whether or not $\mathrm{Col}(A)$ is all of $\mathbb{R}^3$.

This is asking "is there always a solution to $x_1\begin{bmatrix}3\\-1\\5\end{bmatrix}+x_2\begin{bmatrix}4\\6\\14\end{bmatrix}+x_3\begin{bmatrix}-2\\4\\0\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$"

equivalently, is there always a solution to

$$\begin{bmatrix}3 & 4 & -2\\ -1 & 6 & 4\\ 5 & 14 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}?$$

$$\begin{vmatrix}3 & 4 & -2\\ -1 & 6 & 4\\ 5 & 14 & 0\end{vmatrix}=(-2)(-1)^{1+3}\begin{vmatrix}-1 & 6\\ 5 & 14\end{vmatrix}+(4)(-1)^{2+3}\begin{vmatrix}3 & 4\\ 5 & 14\end{vmatrix}+0$$
$$=(-2)(-44)-4(22)$$
$$=0$$

$\therefore$ The equations do not always have solutions (properties of det from last week)

$\therefore$ $\mathrm{col}(A)\neq\mathbb{R}^3$

Example

Following up on the last example, determine whether or not $[5,5,10]^T\in\mathrm{col}(A)$.

is there at least one solution to

$$x_1\begin{bmatrix}3\\-1\\5\end{bmatrix}+x_2\begin{bmatrix}4\\6\\14\end{bmatrix}+x_3\begin{bmatrix}-2\\4\\0\end{bmatrix}=\begin{bmatrix}5\\5\\10\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}3 & 4 & -2 & 5\\ -1 & 6 & 4 & 5\\ 5 & 14 & 0 & 10\end{array}\right]\sim\left[\begin{array}{ccc|c}1 & 0 & -14/11 & 0\\ 0 & 1 & 5/11 & 0\\ 0 & 0 & 0 & 1\end{array}\right]$$

$0\neq1$, the equation has no solution,

$$\therefore\quad\begin{bmatrix}5\\5\\10\end{bmatrix}\notin\mathrm{col}(A)$$

Definition of (and a theorem about) Null Space

The ***null space*** of a matrix A is the set of all solutions to the equation $A\vec{x} = \vec{0}$. If A is an $m \times n$ matrix, then $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

Example

Let $A = \begin{bmatrix} 2 & -2 & 3 \\ 4 & 1 & 6 \end{bmatrix}$. Determine whether or not $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^T$ is an element of either the column space or the null space of A. Also, find a spanning set for $\mathbf{nul}(A)$

$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \notin \mathrm{col}(A)$; the vector has the wrong, fricking, dimensions.

$$x_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + x_3 \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

The vector does have the right dimension for nul(A)

$$\begin{bmatrix} 2 & -2 & 3 \\ 4 & 1 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 - 2 + 0 \\ 8 + 1 + 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \notin \mathrm{nul}(A)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathrm{nul}(A) \Rightarrow \begin{bmatrix} 2 & -2 & 3 \\ 4 & 1 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 0 \\ 4 & 1 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Vectors in the nullspace have form

$$\begin{bmatrix} -\frac{3}{2}x_3 \\ 0 \\ x_3 \end{bmatrix} \text{ which can be written as } x_3 \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad \mathrm{nul}(A) = \mathrm{span}\left\{ \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$