Definition

The ***kernel*** of the linear transformation T is the set of all solutions to the equation $T(\vec{x}) = \vec{0}$.

Example

Suppose that $T: \mathbb{R}^3 \to \mathbb{R}^4$ and that $T(\vec{e}_1) = [2 \quad -4 \quad 1 \quad 2]^T$, $T(\vec{e}_2) = [-1 \quad 2 \quad -2 \quad -7]^T$, and $T(\vec{e}_3) = [3 \quad -6 \quad 1 \quad 1]$. Find the kernel and range of T.

$$T(\vec{x}) = A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & -6 \\ 1 & -2 & 1 \\ 2 & -7 & 1 \end{bmatrix}$$

Finding the kernel of T is the same as finding the null space of A.

$$A\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ -4 & 2 & -6 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -7 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

General solution

$$\begin{cases} x_1 = -\frac{5}{3}x_3 \\ x_2 = -\frac{1}{3}x_3 \\ x_3 \text{ is free} \end{cases}$$

Solution vectors: $x_3 \begin{bmatrix} -5/3 \\ -1/3 \\ 1 \end{bmatrix}$

$\therefore$ The Kernel of T is span $\left\{ \begin{bmatrix} -5/3 \\ -1/3 \\ 1 \end{bmatrix} \right\}$

Check: $T\left(\begin{bmatrix} -5/3 \\ -1/3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & -6 \\ 1 & -2 & 1 \\ 2 & -7 & 1 \end{bmatrix} \begin{bmatrix} -5/3 \\ -1/3 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

Example

Suppose that $T: P_2 \to \mathbb{R}^2$ with the rule $T(\vec{p}) = \begin{bmatrix} \vec{p}(3) \\ \vec{p}(5) \end{bmatrix}$. What is the image of $\vec{p}_1$ where $\vec{p}_1(t) = 2 - t + 2t^2$? What do polynomials in the kernel of T all have in common? Find the transformation matrix for T using (from $\mathbb{R}^3$) the vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ to represent, respectively, the polynomials with formulas "1," "t," and "t^2". Find the null space of the transformation matrix and relate it back to polynomials in the kernel of T.

$$T(\vec{p}_1(t) = 2 - t + 2t^2) = \begin{bmatrix} \vec{p}_1(3) \\ \vec{p}_1(5) \end{bmatrix}$$
$$= \begin{bmatrix} 17 \\ 47 \end{bmatrix}$$

If $T(\vec{p}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $p(3) = 0$ and $p(5) = 0$ which means that $(t-3)$ and $(t-5)$ are both factors of $\vec{p}_1$.

associate $a + bt + ct_2$ with $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then

$$T(\vec{p}(t) = 1) = \begin{bmatrix} \vec{p}(3) \\ \vec{p}(5) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad T(\vec{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T(\vec{p}(t) = t) = \begin{bmatrix} \vec{p}(3) \\ \vec{p}(5) \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}; \quad T(\vec{e}_2) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$T(\vec{p}(t) = t^2) = \begin{bmatrix} \vec{p}(3) \\ \vec{p}(5) \end{bmatrix} = \begin{bmatrix} 9 \\ 25 \end{bmatrix}; \quad T(\vec{e}_3) = \begin{bmatrix} 9 \\ 25 \end{bmatrix}$$

$$\therefore T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = A \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ where } A = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 5 & 25 \end{bmatrix}$$

Reality check

$$\begin{bmatrix} 1 & 3 & 9 \\ 1 & 5 & 25 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + 3b + 9c \\ a + 5b + 25c \end{bmatrix}$$ which, by golly

$$= \begin{bmatrix} p(3) \\ p(5) \end{bmatrix}$$ where $p(t) = a + bt + ct^2$

The Kernel of T consists of solutions to $A\vec{x} = \vec{0}$

$$\begin{array}{ccc} a & b & c \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 9 & 0 \\ 1 & 5 & 25 & 0 \end{array}\right] \sim \left(\begin{array}{ccc|c} 1 & 0 & -15 & 0 \\ 0 & 1 & 8 & 0 \end{array}\right)$$

$$a = 15c$$

$$b = -8c$$

$\therefore$ Polynomials in the kernel satisfy:

$$\begin{aligned} \vec{p}(t) &= a + bt + ct^2 \\ &= 15c - 8ct + ct^2 \\ &= c(15 - 8t + t^2) \\ &= c(t^2 - 8t + 15) \\ &= c(t-3)(t-5) \end{aligned}$$

MATH IS <u>AWESOME</u>!