Let $A = \begin{bmatrix} 2 & 1 & -4 & 3 & -2 & -5 \\ -1 & 1 & 2 & 3 & 1 & -11 \\ 1 & 4 & -2 & -3 & -1 & 11 \\ -2 & -2 & 4 & -1 & 2 & -1 \end{bmatrix}$. Then $A \sim \begin{bmatrix} 1 & 0 & -2 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}$.

What are the dimensions of the null space and column space of A and how do you know. What do these dimensions sum to?

3 The null space is three-dimensional. The RREF(A) has three free variables.

+ 3 The column space is three-dimensional (coincidence!) The RREF(A) has three pivot columns.

6 columns because every column is either a pivot column or associated with a free variable in the solution to $A\vec{x} = \vec{0}$.

dim (nulspace) + dim (column space) = # of columns. ALWAYS!

State bases for the null space and column space of A.

Basis for col (A) $= \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \\ -1 \end{bmatrix} \right\}$

$A\vec{x} = \vec{0}$ has general solution: $\begin{cases} x_1 = 2x_3 + x_5 - 2x_6 \\ x_2 = 0 \\ x_3 \text{ is free} \\ x_4 = 3x_6 \\ x_5 \text{ is free} \\ x_6 \text{ is free} \end{cases}$

So solutions can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ A basis for the null space is $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Determine the dimension of $\operatorname{span}\left(\underbrace{\left\{\begin{bmatrix}1\\-1\\0\\4\\-2\end{bmatrix},\begin{bmatrix}-4\\4\\0\\-16\\8\end{bmatrix},\begin{bmatrix}-1\\2\\1\\3\\-2\end{bmatrix},\begin{bmatrix}1\\0\\1\\11\\-6\end{bmatrix},\begin{bmatrix}-2\\3\\1\\-1\\0\end{bmatrix}\right\}}_{W}\right)$. Explain your thought process!

The dimension of the span is the number of pivot columns in $\underbrace{\begin{bmatrix}1 & -4 & 1 & 1 & -2\\ -1 & 4 & 2 & 0 & 3\\ 0 & 0 & 1 & 1 & 1\\ 4 & -16 & 3 & 11 & -1\\ 2 & 8 & -2 & -6 & 0\end{bmatrix}}_{A}$

$\operatorname{span}(W) = \operatorname{col}(A)$

which is two.

Find a basis for the null space of B where $B=\begin{bmatrix}-i & 1\\ -1 & -i\end{bmatrix}$.

$B\vec{x}=\vec{0}$

$\left[\begin{array}{cc|c}-i & 1 & 0\\ -1 & -i & 0\end{array}\right]$ $iR_1+R_2\to R_2$ $\left[\begin{array}{cc|c}-i & 1 & 0\\ 0 & 0 & 0\end{array}\right]$

Clearly $-ix_1 = -x_2$ (x_2 free)

So $x_1 = \dfrac{-x_2}{-i}\cdot\dfrac{i}{i}$

$= -ix_2$

$\therefore \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_2\begin{bmatrix}-i\\1\end{bmatrix}$

$\left\{\begin{bmatrix}-i\\1\end{bmatrix}\right\}$ for a basis for the nullspace of B.