Bases of Vector Spaces

A set of linearly independent vectors from a vector space whose span includes the entire vector space is called a ***basis*** for the vector space.

Example

Show that the three elementary vectors from $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$.

Example

Show that $\left\{ \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} -3 & -1 & 5 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 1 & 0 & 4 \end{bmatrix}^{\mathrm{T}} \right\}$ forms a basis for $\mathbb{R}^3$.

A theorem about bases of vector spaces

If a vector space has a basis containing exactly n vectors, then any set containing at least $n+1$ vectors is linearly dependent.

Example

Show that if $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for the vector space V, then any three vectors from V must be linearly dependent.

A Theorem about finite dimensional vector spaces

If a vector space has one basis containing exactly n vectors, then every basis of that vector space contains exactly n vectors. We call this number n the dimension of the vector space.

Proof

A plethora of theorems about *n*-dimensional vector spaces

1. Any set of n linearly independent vectors in the space forms a basis for the space.
2. Any set of n vectors that spans the space forms a basis for the space.
3. Any set of linearly independent vectors is a subset of a basis for the space.
4. Any set of vectors that spans the space contains a subset that forms a basis for the space.

Example of theorem 3

Show that $\left\{[1,2,3]^{\mathsf{T}}, [4,5,6]^{\mathsf{T}}\right\}$ is a subset of a basis of $\mathbb{R}^3$.

Finding bases for the null space and column space of a matrix

- A spanning set of the solution set to the homogenous system $A\vec{x}=\vec{0}$ forms a basis for the null space of A.
- The pivot columns of A form a basis for the column space of A.

Example: Let $\mathbf{A}=\begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}$. Then $\mathrm{RREF}(\mathbf{A})=\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for the null space of A and explicitly show that this is indeed a basis for the null space of A.

Example: Let $\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}$. Then $\mathrm{RREF}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for the column space of A and explicitly show that this is indeed a basis for the column space of A.

Example

Show that any two vectors from the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 13 \\ -11 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -19 \\ 23 \end{bmatrix} \right\}$ form a basis for the span of the set.

Example

Consider the set of vectors of form $\begin{bmatrix} s \\ 2s \\ t \end{bmatrix}$; call the set H.

1. Show that H is a subspace of $\mathbb{R}^3$.
2. Every vector is H is in $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Specifically, $\begin{bmatrix} s \\ 2s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Despite this fact, there is no way that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ forms a basis for H. Give two distinct and specific reasons why this is the case.
3. Find a basis for H.