Definition
The ***kernel*** of the linear transformation T is the set of all solutions to the equation $T(\vec{x}) = \vec{0}$.

Example
Suppose that $T:\ \mathbb{R}^3 \to \mathbb{R}^4$ and that $T(\vec{e}_1) = \begin{bmatrix} 2 & -4 & 1 & 2 \end{bmatrix}^{\mathrm{T}}$, $T(\vec{e}_2) = \begin{bmatrix} -1 & 2 & -2 & -7 \end{bmatrix}^{\mathrm{T}}$, and $T(\vec{e}_3) = \begin{bmatrix} 3 & -6 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$. Find the kernel and range of T.

Example

Suppose that $T: P_2 \to \mathbb{R}^2$ with the rule $T(\vec{p}) = \begin{bmatrix} \vec{p}(3) \\ \vec{p}(5) \end{bmatrix}$. What is the image of $\vec{p}_1$ where $\vec{p}_1(t) = 2 - t + 2t^2$? What do polynomials in the kernel of T all have in common? Find the transformation matrix for T using (from $\mathbb{R}^3$) the vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ to represent, respectively, the polynomials with formulas "1," "t," and "t^2". Find the null space of the transformation matrix and relate it back to polynomials in the kernel of T.

Bases of Vector Spaces

A set of linearly independent vectors from a vector space whose span includes the entire vector space is called a ***basis*** for the vector space.

Example

Show that the three elementary vectors from $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$.

Example

Show that $\left\{ \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} -3 & -1 & 5 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 1 & 0 & 4 \end{bmatrix}^{\mathrm{T}} \right\}$ forms a basis for $\mathbb{R}^3$.

A theorem about bases of vector spaces

If a vector space has a basis containing exactly n vectors, then any set containing at least $n+1$ vectors is linearly dependent.

Example

Show that if $\left\{ \vec{b}_1, \vec{b}_2 \right\}$ is a basis for the vector space V, then any three vectors from V must be linearly dependent.

A Theorem about finite dimensional vector spaces

If a vector space has one basis containing exactly n vectors, then every basis of that vector space contains exactly n vectors. We call this number n the dimension of the vector space.

Proof

A plethora of theorems about *n*-dimensional vector spaces

1. Any set of n linearly independent vectors in the space forms a basis for the space.
2. Any set of n vectors that spans the space forms a basis for the space.
3. Any set of linearly independent vectors is a subset of a basis for the space.
4. Any set of vectors that spans the space contains a subset that forms a basis for the space.

Example of theorem 3

Show that $\left\{[1,2,3]^{\mathsf{T}}, [4,5,6]^{\mathsf{T}}\right\}$ is a subset of a basis of $\mathbb{R}^3$.

Finding bases for the null space and column space of a matrix

- A spanning set of the solution set to the homogenous system $A\vec{x}=\vec{0}$ forms a basis for the null space of A.
- The pivot columns of A form a basis for the column space of A.

Example: Let $\mathbf{A}=\begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}$. Then $\mathrm{RREF}(\mathbf{A})=\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for the null space of A and explicitly show that this is indeed a basis for the null space of A.

Example: Let $\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}$. Then $\mathrm{RREF}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for the column space of A and explicitly show that this is indeed a basis for the column space of A.

Example

Show that any two vectors from the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 13 \\ -11 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -19 \\ 23 \end{bmatrix} \right\}$ form a basis for the span of the set.