Definition

The ***kernel*** of the linear transformation T is the set of all solutions to the equation $T(\vec{x}) = \vec{0}$.

Example

Suppose that $T: \mathbb{R}^3 \to \mathbb{R}^4$ and that $T(\vec{e}_1) = [2 \quad -4 \quad 1 \quad 2]^{\mathrm{T}}$, $T(\vec{e}_2) = [-1 \quad 2 \quad -2 \quad -7]^{\mathrm{T}}$, and $T(\vec{e}_3) = [3 \quad -6 \quad 1 \quad 1]^{\mathrm{T}}$. Find the kernel and range of T.

Let $T(\vec{x}) = A\vec{x}$. Then $\mathrm{Ker}(T) = \mathrm{nul}(A)$ and $\mathrm{range}(T) = \mathrm{col}(A)$.

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & -6 \\ 1 & -2 & 1 \\ 2 & -7 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ -4 & 2 & -6 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -7 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Fact 1: The general solution to $A\vec{x} = \vec{0}$ is $\begin{cases} x_1 = -\frac{5}{3}x_3 \\ x_2 = -\frac{1}{3}x_3 \\ x_3 \text{ is free} \end{cases}$

$\therefore$ The null space is the span of $\left\{ \begin{bmatrix} -5/3 \\ -1/3 \\ 1 \end{bmatrix} \right\}$

$\boxed{\text{The pivot columns of } A \text{ span the column space of } A.}$

$\therefore \mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ -4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \\ -7 \end{bmatrix} \right\}$.

This column space is a two-dimensional subspace of $\mathbb{R}^4$!

Example

Suppose that $T: P_2 \to \mathbb{R}^2$ with the rule $T(\vec{p}) = \begin{bmatrix} \vec{p}(3) \\ \vec{p}(5) \end{bmatrix}$. (a) What is the image of $\vec{p}_1$ where $\vec{p}_1(t) = 2 - t + 2t^2$? (b) What do polynomials in the kernel of T all have in common? (c) Find the transformation matrix for T using (from $\mathbb{R}^3$) the vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ to represent, respectively, the polynomials with formulas "1," "t," and "t^2". Find the null space of the transformation matrix and relate it back to polynomials in the kernel of T.

(a) $$T(2 - t + 2t^2) = \begin{bmatrix} 2 - 3 + 2(3)^2 \\ 2 - 5 + 2(5)^2 \end{bmatrix} = \begin{bmatrix} 17 \\ 47 \end{bmatrix}$$

(b) $$\vec{p} \in Ker(T) \Rightarrow T(\vec{p}) = \vec{0} \Rightarrow \begin{cases} \vec{p}(3) = 0 \\ \vec{p}(5) = 0 \end{cases}$$

The zeros of $\vec{p}$ are 3 and 5.

(c) $(a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3)$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a \cdot 1 + bt + ct^2$$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longleftrightarrow \vec{p}(t) = 1$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longleftrightarrow \vec{p}(t) = t$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longleftrightarrow \vec{p}(t) = t^2$

$$T(1) = \begin{bmatrix} \vec{p}(3) \\ \vec{p}(5) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ where $\underbrace{\vec{p}(t) = 1}_{\text{constant function}}$

The output is always 1.

$$T(t) = \begin{bmatrix} \vec{p}(3) \\ \vec{p}(5) \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$ where $\vec{p}(t) = t$

$$T(t^2) = \begin{bmatrix} \vec{p}(3) \\ \vec{p}(5) \end{bmatrix} = \begin{bmatrix} 9 \\ 25 \end{bmatrix}$$ where $\vec{p}(t) = t^2$

$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $T(\vec{e}_3) = \begin{bmatrix} 9 \\ 25 \end{bmatrix}$

$\therefore$ $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 5 & 25 \end{bmatrix}$

So... $$T(2 - t + 2t^2) = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 5 & 25 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 - 3 + 2 \cdot 3^2 \\ 2 \cdot 1 - 5 + 2 \cdot 5^2 \end{bmatrix}$$

$\begin{pmatrix} \text{constant} \\ \text{linear} \\ \text{quadratic} \end{pmatrix}$

Top Tenner

Bases of Vector Spaces

A set of linearly independent vectors from a vector space whose span includes the entire vector space is called a ***basis*** for the vector space.

Example

Show that the three elementary vectors from $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$.

$\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ is obviously linearly independent,

$\forall \begin{bmatrix}a\\b\\c\end{bmatrix} \in \mathbb{R}^3$, $\begin{bmatrix}a\\b\\c\end{bmatrix} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix}$, so

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ spans $\mathbb{R}^3$. QED

Example

Show that $\left\{ [2 \quad 4 \quad 1]^T, [-3 \quad -1 \quad 5]^T, [1 \quad 0 \quad 4]^T \right\}$ forms a basis for $\mathbb{R}^3$.

We need to show that the set is linear independent and spans $\mathbb{R}^3$. Theorem 8 to the rescue..

This is akin to properties e & h applied to the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & -1 & 0 \\ 1 & 5 & 4 \end{bmatrix}$. $\det(A) = 61 \neq 0$ (properties m)

QED

> **A theorem about bases of vector spaces**
>
> If a vector space has a basis containing exactly n vectors, then any set containing at least $n+1$ vectors is linearly dependent.

Example

Show that if $\{\vec{b}_1, \vec{b}_2\}$ is a basis for the vector space V, then any three vectors from V must be linearly dependent.

Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subset V$. Since $\{\vec{b}_1, \vec{b}_2\}$ spans V, $\exists$ constants, $x_{ij} \in \mathbb{R}$, such that

$$x_{11}\vec{b}_1 + x_{12}\vec{b}_2 = \vec{v}_1$$
$$x_{21}\vec{b}_1 + x_{22}\vec{b}_2 = \vec{v}_2$$
and
$$x_{31}\vec{b}_1 + x_{32}\vec{b}_2 = \vec{v}_3$$

Suppose that $y_1\vec{v}_1 + y_2\vec{v}_2 + y_3\vec{v}_3 = \vec{0}$ for scalars y_1, y_2, y_3

Note: For given $\vec{v}_1, \vec{v}_2, \vec{v}_3$, $x_{11} - x_{32}$ are constants, they're not variables. The variables are y_1, y_2, y_3 - that's the equation we're solving.

$$y_1\vec{v}_1 + y_2\vec{v}_2 + y_3\vec{v}_3 = \vec{0} \Rightarrow y_1(x_{11}\vec{b}_1 + x_{12}\vec{b}_2) + y_2(x_{21}\vec{b}_1 + x_{22}\vec{b}_2) + y_3(x_{31}\vec{b}_1 + x_{32}\vec{b}_2) = \vec{0}$$

$$\Rightarrow (x_{11}y_1 + x_{21}y_2 + x_{31}y_3)\vec{b}_1 + (x_{12}y_1 + x_{22}y_2 + x_{32}y_3)\vec{b}_2 = \vec{0}$$

$\{\vec{b}_1, \vec{b}_2\}$ is a linearly independent set, so

$$\begin{cases} x_{11}y_1 + x_{21}y_2 + x_{31}y_3 = 0 \\ x_{12}y_1 + x_{22}y_2 + x_{32}y_3 = 0 \end{cases} \qquad \text{QED}$$

This system has more variables (y_1, y_2, y_3), then equations so it cannot have a unique solution

The most important theorem in linear algebra

A Theorem about finite dimensional vector spaces

If a vector space has one basis containing exactly n vectors, then every basis of that vector space contains exactly n vectors. We call this number n the dimension of the vector space.

Proof Proof by contradiction

Assume that there are bases with two different number of vectors (m vectors and n vectors). $m \neq n$

If $m < n$, then the set with n vectors cannot possibly be linearly independent (last page)

If $n < m$, then the set with m vectors cannot possibly be linearly independent

A plethora of theorems about *n*-dimensional vector spaces

1. Any set of n linearly independent vectors in the space forms a basis for the space.
2. Any set of n vectors that spans the space forms a basis for the space.
3. Any set of linearly independent vectors is a subset of a basis for the space.
4. Any set of vectors that spans the space contains a subset that forms a basis for the space.

Example of theorem 3

Show that $\left\{[1,2,3]^{\mathrm{T}},[4,5,6]^{\mathrm{T}}\right\}$ is a subset of a basis of $\mathbb{R}^3$.

Bases for $\mathbb{R}^3$ have three vectors.

$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$ is obviously linearly independent, so by Plethora Theorem, we can build it up to a basis for $\mathbb{R}^3$.

$$\left\{\underset{\vec{v}_1}{\begin{bmatrix}1\\2\\3\end{bmatrix}},\ \underset{\vec{v}_2}{\begin{bmatrix}4\\5\\6\end{bmatrix}},\ \begin{bmatrix}v_{11}+v_{21}\\v_{12}+v_{22}\\v_{13}\ \text{not}+v_{23}\end{bmatrix}\right\}$$

vector 1 vector 2

The set $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}5\\7\\3\end{bmatrix}\right\}$ forms basis for $\mathbb{R}^3$.

$$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\}\subset\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{pmatrix}5\\7\\3\end{pmatrix}\right\}$$

QED

Finding bases for the null space and column space of a matrix

- A spanning set of the solution set to the homogenous system $A\vec{x}=\vec{0}$ forms a basis for the null space of A.
- The pivot columns of A form a basis for the column space of A.

Example: Let $\mathbf{A}=\begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}$. Then $\operatorname{RREF}(\mathbf{A})=\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for the null space of A and explicitly show that this is indeed a basis for the null space of A.

From the RREF(A) we can infer:

$$\begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases}$$

[The augmented matrix for $A\vec{x}=\vec{0}$ is

$$\left[\begin{array}{cccc|c} 2 & 1 & 3 & 3 & 0 \\ 1 & -1 & 3 & 1 & 0 \\ -2 & 1 & -5 & 3 & 0 \\ 3 & 2 & 4 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$]

$\therefore$ Solutions to $A\vec{x}=\vec{0}$ can be written

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for null(A)

Check: $$\begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4+1+3+0 \\ -2-1+3+0 \\ 4+1-5+0 \\ -6+2+4+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

e.g. no vector of form $\begin{bmatrix} * \\ * \\ * \\ \text{not zero } * \end{bmatrix}$ is in the nullspace

Pivot Columns

Pivot Entries

Example: Let $\mathbf{A}=\begin{bmatrix}2&1&3&3\\1&-1&3&1\\-2&1&-5&3\\3&2&4&2\end{bmatrix}$. Then $\mathrm{RREF}(\mathbf{A})=\begin{bmatrix}1&0&2&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$.

Find a basis for the column space of A and explicitly show that this is indeed a basis for the column space of A.

A basis for the column space of A is:

$$\left\{\begin{bmatrix}2\\1\\-2\\3\end{bmatrix},\begin{bmatrix}1\\-1\\1\\2\end{bmatrix},\begin{bmatrix}3\\1\\3\\2\end{bmatrix}\right\}$$

basis for col(A)

Show that this is a basis

$$x_1\begin{bmatrix}2\\1\\-2\\3\end{bmatrix}+x_2\begin{bmatrix}1\\-1\\1\\2\end{bmatrix}+x_3\begin{bmatrix}3\\1\\3\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}2&1&3&0\\1&-1&1&0\\-2&1&3&0\\3&2&2&0\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right]\Rightarrow\begin{cases}x_1=0\\x_2=0\\x_3=0\end{cases}$$

$\therefore$ The set is linearly independent.

We also need to show that the three vectors span the column space.

We need to show that anything that can be written using all four columns can also be written using only columns 1, 2, and 4. This can be done by showing that C_3 is a linear combination of the other three columns.

$$x_1\begin{bmatrix}2\\1\\-2\\3\end{bmatrix}+x_2\begin{bmatrix}1\\-1\\1\\2\end{bmatrix}+x_3\begin{bmatrix}3\\1\\3\\2\end{bmatrix}=\begin{bmatrix}3\\3\\-5\\4\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}2&1&3&3\\1&-1&1&3\\-2&1&3&-5\\3&2&2&4\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&0&2\\0&1&0&-1\\0&0&1&0\\0&0&0&0\end{array}\right]\quad\therefore\ 2\begin{bmatrix}2\\1\\-2\\3\end{bmatrix}+(-1)\begin{bmatrix}1\\-1\\1\\2\end{bmatrix}+0\begin{bmatrix}3\\1\\3\\2\end{bmatrix}=\begin{bmatrix}3\\3\\-5\\4\end{bmatrix}$$

QED

Any column that is not a pivot column is a linear combination of the columns that preceded it.

Example

Show that any two vectors from the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 13 \\ -11 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -19 \\ 23 \end{bmatrix} \right\}$ form a basis for the span of the set.

Call the set W.

$$\underbrace{\begin{bmatrix} 2 & 3 & 0 & 11 & 9 \\ 3 & -2 & 13 & 10 & -19 \\ -1 & 4 & -11 & 0 & 23 \end{bmatrix}}_{\text{call this } A} \sim \begin{bmatrix} 1 & 0 & 3 & 4 & -3 \\ 0 & 1 & -2 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This shows that the first two columns of A form a basis for Col(A) and, as such, also form a basis for the span of W.

$\therefore$ Any two linearly independent vectors from W form a basis for W. The vectors in W are, by inspection, pair-wise linearly independent. (no two are multiples).

QED