Definition

The ***kernel*** of the linear transformation T is the set of all solutions to the equation $T(\vec{x}) = \vec{0}$.

Example

Suppose that $T: \mathbb{R}^3 \to \mathbb{R}^4$ and that $T(\vec{e}_1) = [2 \quad -4 \quad 1 \quad 2]^{\mathrm{T}}$, $T(\vec{e}_2) = [-1 \quad 2 \quad -2 \quad -7]^{\mathrm{T}}$, and $T(\vec{e}_3) = [3 \quad -6 \quad 1 \quad 1]^{\mathrm{T}}$. Find the kernel and range of T.

$$T(\vec{x}) = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & -6 \\ 1 & -2 & 1 \\ 2 & -7 & 1 \end{bmatrix} \vec{x}$$ (call the matrix A), so the

Kernel of T is the null space of A.

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ -4 & 2 & -6 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -7 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

gen solution: $\begin{cases} x_1 = \frac{-5}{3} x_3 \\ x_2 = \frac{-1}{3} x_3 \\ x_3 \text{ is free} \end{cases}$

A basis for null(A) and, consequently, Ker(T) is

$$\left\{ \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix} \right\}.$$

$\vec{b} \in$ Range of T iff $\exists\, x_1, x_2, x_3 \ni x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + x_3 T(\vec{e}_3) = \vec{b}$.

This is analogous to saying that $\vec{b}$ is in the column space of A. We established above that the pivot columns of A are $\vec{c}_1$ and $\vec{c}_2$.

$\therefore$ A basis for range(A) is

$$\left\{ \begin{bmatrix} 2 \\ -4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \\ -7 \end{bmatrix} \right\}$$

Example

Suppose that $T : P_2 \to \mathbb{R}^2$ with the rule $T(\vec{p}) = \begin{bmatrix} \vec{p}(3) \\ \vec{p}(5) \end{bmatrix}$. What is the image of $\vec{p}_1$ where $\vec{p}_1(t) = 2 - t + 2t^2$? What do polynomials in the kernel of T all have in common? Find the transformation matrix for T using (from $\mathbb{R}^3$) the vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ to represent, respectively, the polynomials with formulas "1," "t," and "t^2". Find the null space of the transformation matrix and relate it back to polynomials in the kernel of T.

Bases of Vector Spaces

A set of linearly independent vectors from a vector space whose span includes the entire vector space is called a ***basis*** for the vector space.

Example

Show that the three elementary vectors from $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$.

① Obviously the only solution to $x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ is the trivial solution.

② $\forall \begin{bmatrix}a\\b\\c\end{bmatrix} \in \mathbb{R}^3$, $\begin{bmatrix}a\\b\\c\end{bmatrix} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix}$

So $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is linearly independent (①), and spans $\mathbb{R}^3$ ②

QED

Example

Show that $\left\{[2 \quad 4 \quad 1]^T, [-3 \quad -1 \quad 5]^T, [1 \quad 0 \quad 4]^T\right\}$ forms a basis for $\mathbb{R}^3$.

Let $A = \begin{bmatrix}2 & -3 & 1\\4 & -1 & 0\\1 & 5 & 4\end{bmatrix}$

$$\det(A) = (1)(-1)^4\begin{vmatrix}4 & -1\\1 & 5\end{vmatrix} + 0C_{23} + 4(-1)^6\begin{vmatrix}2 & -3\\4 & -1\end{vmatrix}$$

$$= (20-(-1)) + 4(-2-(-12))$$

$$= 61$$

$$\neq 0$$

Since property m of the given theorem is true for A, so are properties e (linear independence of columns) and h (columns span $\mathbb{R}^3$)

QED

A theorem about bases of vector spaces

If a vector space has a basis containing exactly n vectors, then any set containing at least $n+1$ vectors is linearly dependent.

Example

Show that if $\{\vec{b}_1, \vec{b}_2\}$ is a basis for the vector space V, then any three vectors from V must be linearly dependent.

Assume that $\{\vec{b}_1, \vec{b}_2\}$ forms a basis for V and that $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in V$. Then:

(The basis spans V)

$\exists$ scalars $x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}$ such that

$\vec{v}_1 = x_{11}\vec{b}_1 + x_{12}\vec{b}_2$, $\vec{v}_2 = x_{21}\vec{b}_1 + x_{22}\vec{b}_2$, and $\vec{v}_3 = x_{31}\vec{b}_1 + x_{32}\vec{b}_2$.

Consider the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. This is equivalent to

$$c_1(x_{11}\vec{b}_1 + x_{12}\vec{b}_2) + c_2(x_{21}\vec{b}_1 + x_{22}\vec{b}_2) + c_3(x_{31}\vec{b}_1 + x_{32}\vec{b}_2) = \vec{0}$$

(Focus: The unknown here are c_1, c_2, c_3)

$$(x_{11}c_1 + x_{21}c_2 + x_{31}c_3)\vec{b}_1 + (x_{12}c_1 + x_{22}c_2 + x_{32}c_3)\vec{b}_2 = \vec{0}$$

(The x_{ij}'s are the coefficients, the c_i's are what we're solving for.)

Bases are linearly independent sets so it follows that

$$\begin{cases} x_{11}c_1 + x_{21}c_2 + x_{31}c_3 = 0 \\ x_{12}c_1 + x_{22}c_2 + x_{23}c_3 = 0 \end{cases}$$

This is a system of two equations and three unknowns (c_1, c_2, c_3)

So it cannot have exactly one solution. This system is

homogenous, so it has at least one solution ($c_1 = c_2 = c_3 = 0$). The only option left is that the system has an unlimited number of solutions. Q.E.D.

A Theorem about finite dimensional vector spaces

If a vector space has one basis containing exactly n vectors, then every basis of that vector space contains exactly n vectors. We call this number n the dimension of the vector space.

Proof

Suppose that V has bases containing, respectively, m and n vectors.

If $m < n$, the set of vectors containing n vectors must be linearly dependent. Likewise, if $n < m$ the set containing m vectors must be linearly dependent.

$\therefore \quad m = n$

A plethora of theorems about *n*-dimensional vector spaces

1. Any set of n linearly independent vectors in the space forms a basis for the space.
2. Any set of n vectors that spans the space forms a basis for the space.
3. Any set of linearly independent vectors is a subset of a basis for the space.
4. Any set of vectors that spans the space contains a subset that forms a basis for the space.

Example of theorem 3

Show that $\left\{[1,2,3]^{T},[4,5,6]^{T}\right\}$ is a subset of a basis of $\mathbb{R}^3$.

$\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}4\\5\\6\end{bmatrix}$ are obviously linearly independent.

Let's mess up $\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}4\\5\\6\end{bmatrix}$ and get $\begin{bmatrix}5\\7\\34\end{bmatrix}$

$$\begin{vmatrix}1 & 4 & 5\\2 & 5 & 7\\3 & 6 & 34\end{vmatrix} = -75 \neq 0 \quad \therefore \left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}5\\7\\34\end{bmatrix}\right\}$$

is linearly independent. $\therefore$ The set forms a basis for $\mathbb{R}^3$ (#1 above).

Finding bases for the null space and column space of a matrix

- A spanning set of the solution set to the homogenous system $A\vec{x} = \vec{0}$ forms a basis for the null space of A.
- The pivot columns of A form a basis for the column space of A.

Example: Let $\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}$. Then $\mathrm{RREF}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. $\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$

Find a basis for the null space of A and explicitly show that this is indeed a basis for the null space of A.

The general solution to $A\vec{x} = \vec{0}$ is $\begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases}$

Vectors in the null space can be written $x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

Ergo, $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ spans $\mathrm{nul}(A)$. $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is self-evidently linearly independent.

$\therefore$ $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ forms a basis for $\mathrm{nul}(A)$.

Check: $\begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

Columns that have leading entries sprang from pivot columns.

Leading entries

Example: Let $\mathbf{A}=\begin{array}{cccc} \downarrow & \downarrow & & \downarrow \\ \end{array}\begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}$. Then $\mathrm{RREF}(\mathbf{A})=\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for the column space of A and explicitly show that this is indeed a basis for the column space of A.

A basis for col(A) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}$

(1) we need to show $\mathrm{span}(\{\vec{c}_1, \vec{c}_2, \vec{c}_4\}) = \mathrm{span}(\{\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4\})$

we can do this by showing that $\vec{c}_3 \in \mathrm{span}(\{\vec{c}_1, \vec{c}_2, \vec{c}_4\})$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 1 & 3 \\ -2 & 1 & 3 & -5 \\ 3 & 2 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \therefore 2\begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -5 \\ 4 \end{bmatrix}.$$

(2) we need to show $\{\vec{c}_1, \vec{c}_2, \vec{c}_4\}$ is linearly independent.

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 1 & 0 \\ -2 & 1 & 3 & 0 \\ 3 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

ergo, the only solution to $x_1\vec{c}_1 + x_2\vec{c}_2 + x_3\vec{c}_4 = \vec{0}$ is indeed $\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0. \end{cases}$

$\therefore$ $\{\vec{c}_1, \vec{c}_2, \vec{c}_4\}$ is linearly independent and spans col(A); i.e. the set is a basis for col(A)

Example

Show that any two vectors from the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 13 \\ -11 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -19 \\ 23 \end{bmatrix} \right\}$ form a basis for the span of the set.

$$\left[\begin{array}{ccccc|c} 2 & 3 & 0 & 11 & 9 & 0 \\ 3 & -2 & 13 & 10 & -19 & 0 \\ -1 & 4 & -11 & 0 & 23 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & -2 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ The general solution to $x_1 \vec{C}_1 + x_2 \vec{C}_2 + x_3 \vec{C}_3 + x_4 \vec{C}_4 + x_5 \vec{C}_5 = \vec{0}$

is $\begin{cases} x_1 = -3x_3 - 4x_4 + 3x_5 \\ x_2 = 2x_3 - x_4 - 5x_5 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases}$ and $\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

is a basis for the null space which has <u>nothing</u> to do with this question.

This question is about the <u>column</u> <u>space</u> of that matrix which has a basis formed by the pivot columns:

A basis is $\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \right\}$. The five given vectors are obviously pair-wise linearly independent.

Any two linearly independent vectors for 2-dimensional V form a basis for V.