Definition
The ***kernel*** of the linear transformation T is the set of all solutions to the equation $T(\vec{x}) = \vec{0}$.

Example
Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^4$ and that $T(\vec{e}_1) = [2 \quad -4 \quad 1 \quad 2]^T$, $T(\vec{e}_2) = [-1 \quad 2 \quad -2 \quad -7]^T$, and $T(\vec{e}_3) = [3 \quad -6 \quad 1 \quad 1]^T$. Find the kernel and range of T.

$T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & -6 \\ 1 & -2 & 1 \\ 2 & -7 & 1 \end{bmatrix}$

$A \sim \begin{bmatrix} 1 & 0 & 5/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ The solution set to

$A\vec{x} = 0$ is $\left\{ \begin{bmatrix} -5/3\, t \\ -1/3\, t \\ t \end{bmatrix} \right\}$

$\therefore \quad \text{Ker}(T) = \text{span}\left\{ \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix} \right\}$

$\left[\begin{array}{cc|c} 2 & -1 & 3 \\ -4 & 2 & -6 \\ 1 & -2 & 1 \\ 2 & -7 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 5/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ This tells us that

$\begin{bmatrix} 3 \\ -6 \\ 1 \\ 1 \end{bmatrix} = \frac{5}{3}\begin{bmatrix} 2 \\ -4 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -2 \\ -7 \end{bmatrix}$

The point being, $\vec{C}_3$ does not expand the span of $\{\vec{C}_1, \vec{C}_2\}$

$\therefore \quad \text{range}(T) = \text{span}(\{\vec{c}_1, \vec{c}_2\})$

Example

Suppose that $T: P_2 \to \mathbb{R}^2$ with the rule $T(\vec{p}) = \begin{bmatrix} \vec{p}(3) \\ \vec{p}(5) \end{bmatrix}$. What is the image of $\vec{p}_1$ where $\vec{p}_1(t) = 2 - t + 2t^2$? What do polynomials in the kernel of T all have in common? Find the transformation matrix for T using (from $\mathbb{R}^3$) the vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ to represent, respectively, the polynomials with formulas "1," "t," and "t^2". Find the null space of the transformation matrix and relate it back to polynomials in the kernel of T.

How does T work?

$$T(2 - t + 2t^2) = \begin{bmatrix} 2 - 3 + 2 \cdot 3^2 \\ 2 - 5 + 2 \cdot 5^2 \end{bmatrix} = \begin{bmatrix} 17 \\ 47 \end{bmatrix}$$

Polynomials in the Ker(T) all map to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which $\vec{p}(3) = 0$ and $\vec{p}(5) = 0$

So, $\vec{p} \in \text{Ker}(T)$ iff $\vec{p}(t) = k(t-3)(t-5)$

$$= kt^2 - 8kt + 15k$$
$$= k(15 - 8t + t^2)$$

$T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T(t) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, and $T(t^2) = \begin{bmatrix} 9 \\ 25 \end{bmatrix}$

$\therefore \quad A = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 5 & 25 \end{bmatrix}$

$\vec{p}(3)$
$\vec{p}(5)$

Exploration

$\vec{p}(t) = 7 - t + 4t^2$; $T(\vec{p}(t)) = \begin{bmatrix} 7 - 3 + 4 \cdot 9 \\ 7 - 5 + 4 \cdot 25 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 1 & 5 & 25 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 9 & 0 \\ 1 & 5 & 25 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -15 & 0 \\ 0 & 1 & 8 & 0 \end{array}\right]$$

So the null space $= \left\{ k \begin{bmatrix} 15 \\ -8 \\ 1 \end{bmatrix} \middle| \, k \in \mathbb{R} \right\}$

Bases of Vector Spaces

A set of linearly independent vectors from a vector space whose span includes the entire vector space is called a ***basis*** for the vector space.

Example

Show that the three elementary vectors from $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$.

Obviously, the only solution to $x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ is $\begin{bmatrix}0\\0\\0\end{bmatrix}$, so $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is linearly independent.

Also, $\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} = u_1\begin{bmatrix}1\\0\\0\end{bmatrix} + u_2\begin{bmatrix}0\\1\\0\end{bmatrix} + u_3\begin{bmatrix}0\\0\\1\end{bmatrix}$, so $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ spans $\mathbb{R}^3$ QED

Example

Show that $\left\{[2 \quad 4 \quad 1]^{\mathrm{T}}, [-3 \quad -1 \quad 5]^{\mathrm{T}}, [1 \quad 0 \quad 4]^{\mathrm{T}}\right\}$ forms a basis for $\mathbb{R}^3$.

We need to show that $A\vec{x} = \vec{b}$ always has a solution where $A = \begin{bmatrix}2 & -3 & 1\\4 & -1 & 0\\1 & 5 & 4\end{bmatrix}$ which we can do by showing that $\det(A) \neq 0$.

$$\det(A) = \sum_{j=1}^{3}[a_{2,j}\, C_{2,j}]$$

$$= 4(-1)^3\begin{vmatrix}-3 & 1\\5 & 4\end{vmatrix} + (-1)(-1)^4\begin{vmatrix}2 & 1\\1 & 4\end{vmatrix} + 0$$

$$= -4(-12-5) - (8-1)$$

$$\neq 0 \qquad \text{QED}$$

A theorem about bases of vector spaces
If a vector space has a basis containing exactly n vectors, then any set containing at least $n+1$ vectors is linearly dependent.

Example
Show that if $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for the vector space V, then any three vectors from V must be linearly dependent.

Let $\vec{v}_1 = k_1\vec{b}_1 + k_2\vec{b}_2$, $\vec{v}_2 = l_1\vec{b}_1 + l_2\vec{b}_2$, and $\vec{v}_3 = m_1\vec{b}_1 + m_2\vec{b}_2$. (I know these scalars exist because $\{\vec{b}_1, \vec{b}_2\}$ spans V).

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

$$\Rightarrow x_1(k_1\vec{b}_1 + k_2\vec{b}_2) + x_2(l_1\vec{b}_1 + l_2\vec{b}_2) + x_3(m_1\vec{b}_1 + m_2\vec{b}_2) = \vec{0}$$

(the unknowns are x_1, x_2, x_3)

$$\Rightarrow (k_1x_1 + l_1x_2 + m_1x_3)\vec{b}_1 + (k_2x_1 + l_2x_2 + m_2x_3)\vec{b}_2 = \vec{0}$$

$$\Rightarrow \begin{cases} k_1x_1 + l_1x_2 + m_1x_3 = 0 \\ k_2x_1 + l_2x_2 + m_2x_3 = 0 \end{cases}$$ ($\vec{b}_1$ & $\vec{b}_2$ are linearly independent)

This system has fewer equations than it has unknowns, so it <u>cannot</u> have a unique solution. The system is homogeneous, so it cannot have no solution.

$\therefore$ The system has an unlimited number of solutions; $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ has nontrivial solutions; $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent.

dimension = number of vectors in the basis

A Theorem about finite dimensional vector spaces

If a vector space has one basis containing exactly n vectors, then every basis of that vector space contains exactly n vectors. We call this number n the dimension of the vector space.

Proof Proof by contradiction. Suppose that there are basis of unequal dimension, m and n.

If $m > n$, since a basis has n vectors, the set with m vectors must be linearly dependent.

Likewise if $n > m$, the set with n vectors must be linearly dependent. ↯

A plethora of theorems about *n*-dimensional vector spaces

1. Any set of n linearly independent vectors in the space forms a basis for the space.
2. Any set of n vectors that spans the space forms a basis for the space.
3. Any set of linearly independent vectors is a subset of a basis for the space.
4. Any set of vectors that spans the space contains a subset that forms a basis for the space.

Example of theorem 3

Show that $\left\{[1,2,3]^{\mathsf{T}},[4,5,6]^{\mathsf{T}}\right\}$ is a subset of a basis of $\mathbb{R}^3$.

If we come up with a third vector, $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, such that $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right\}$ is linearly independent, we will have acheived our mandate.

Let's screw up $2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

$$\begin{bmatrix} 18 \\ 24 \\ 17 \end{bmatrix} \notin \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right\}.$$

screw up

Obviously, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ are linearly independent.

Ergo, the vectors are linearly independent and form a basis for $\mathbb{R}^3$.

Finding bases for the null space and column space of a matrix

- A spanning set of the solution set to the homogenous system $A\vec{x} = \vec{0}$ forms a basis for the null space of A.
- The pivot columns of A form a basis for the column space of A.

Example: Let $\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}$. Then $\mathrm{RREF}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for the null space of A and explicitly show that this is indeed a basis for the null space of A.

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_2 - x_3 = 0 \\ x_4 = 0 \end{cases} \quad \therefore \quad \mathrm{nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

So a basis for nul(A) is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} -2t \\ t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -4t + t + 3t + 0 \\ -2t - t + 3t + 0 \\ 4t + t - 5t + 0 \\ -6t + 2t + 4t + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

Also, we know it only takes one vector to span nul(A) because there was only one free variable associated with rref(A).

dim(null) = # of free variables

dim(col) = # of pivot columns

Ergo

dim(null)+dim(col) = # of columns

Pivot columns

Leading entries

Example: Let $\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}$. Then $\operatorname{RREF}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for the column space of A and explicitly show that this is indeed a basis for the column space of A.

A basis for col(A) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}$.

Let's first establish linear independence.

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 1 & 0 \\ -2 & 1 & 3 & 0 \\ 3 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$ so the only solution to $x_1 \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 1 & 3 \\ -2 & 1 & 3 & -5 \\ 3 & 2 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \begin{bmatrix} 3 \\ 3 \\ -5 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix},$$

so $\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}$ spans col (A)

QED.

Example

Show that any two vectors from the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 13 \\ -11 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -19 \\ 23 \end{bmatrix} \right\}$ form a basis for the span of the set.

The span of the set is equivalent to the column space of

$$A = \begin{bmatrix} 2 & 3 & 0 & 11 & 9 \\ 3 & -2 & 13 & 10 & -19 \\ -1 & 4 & -11 & 0 & 23 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 0 & 3 & 4 & -3 \\ 0 & 1 & -2 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The dimension of the span of the set is two because A has two pivot columns.

By inspection, any pair of vectors in the set is linearly independent.

$\therefore$ Any pair forms a basis for the span of the set.