Bases of Vector Spaces

A set of linearly independent vectors from a vector space whose span includes the entire vector space is called a ***basis*** for the vector space.

Example

Show that the three elementary vectors from $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$.

Obviously, the only solution to $x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

is $x_1=0,\ x_2=0,\ x_3=0$; so, $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are linearly independent.

Obviously, $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$;

So $\vec{e}_1, \vec{e}_2, \vec{e}_3$ span $\mathbb{R}^3$.. QED

Example

Show that $\left\{\begin{bmatrix}2 & 4 & 1\end{bmatrix}^T, \begin{bmatrix}-3 & -1 & 5\end{bmatrix}^T, \begin{bmatrix}1 & 0 & 4\end{bmatrix}^T\right\}$ forms a basis for $\mathbb{R}^3$.

We need to show that the only solution to $\begin{bmatrix}2 & -3 & 1\\4 & -1 & 0\\1 & 5 & 4\end{bmatrix}\vec{x}=\vec{0}$ is $\vec{0}$ and that

$\begin{bmatrix}2 & -3 & 1\\4 & -1 & 0\\1 & 5 & 4\end{bmatrix}\vec{x}=\vec{b}$ has a solution $\forall\ \vec{b}\in\mathbb{R}^3$

$$\begin{vmatrix}2 & -3 & 1\\4 & -1 & 0\\1 & 5 & 4\end{vmatrix}=(4)(-1)^{2+1}\begin{vmatrix}-3 & 1\\5 & 4\end{vmatrix}+(-1)(-1)^{2+2}\begin{vmatrix}2 & 1\\1 & 4\end{vmatrix}+0$$

$$=(-4)(-17)+(-1)(7)$$

$$\neq 0 \qquad \text{QED}$$

A theorem about bases of vector spaces
If a vector space has a basis containing exactly n vectors, then any set containing at least $n+1$ vectors is linearly dependent.

Example
Show that if $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for the vector space V, then any three vectors from V must be linearly dependent.

Let $\vec{v}_1 = k_1\vec{b}_1 + k_2\vec{b}_2$, $\vec{v}_2 = \ell_1\vec{b}_1 + \ell_2\vec{b}_2$, and $\vec{v}_3 = m_1\vec{b}_1 + m_2\vec{b}_2$. (I know these scalars exist because $\{\vec{b}_1, \vec{b}_2\}$ spans V).

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

$$\Rightarrow x_1(k_1\vec{b}_1 + k_2\vec{b}_2) + x_2(\ell_1\vec{b}_1 + \ell_2\vec{b}_2) + x_3(m_1\vec{b}_1 + m_2\vec{b}_2) = \vec{0}$$

(the unknowns are x_1, x_2, x_3)

$$\Rightarrow (k_1x_1 + \ell_1x_2 + m_1x_3)\vec{b}_1 + (k_2x_1 + \ell_2x_2 + m_2x_3)\vec{b}_2 = \vec{0}$$

$$\Rightarrow \begin{cases} k_1x_1 + \ell_1x_2 + m_1x_3 = 0 \\ k_2x_1 + \ell_2x_2 + m_2x_3 = 0 \end{cases}$$ ($\vec{b}_1$ & $\vec{b}_2$ are linearly independent)

This system has fewer equations than it has unknowns, so it cannot have a unique solution. The system is homogeneous, so it cannot have no solution.

$\therefore$ The system has an unlimited number of solutions; $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ has nontrivial solutions; $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent.

$\therefore$ The only solution to
$$x_3\begin{bmatrix}1\\2\\3\end{bmatrix} + x_4\begin{bmatrix}4\\5\\6\end{bmatrix} + x_5\vec{v} = \vec{0}$$
is $x_3 = x_4 = x_5 = 0$.
Three linearly independent vectors from $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$. QED

A Theorem about finite dimensional vector spaces

If a vector space has one basis containing exactly n vectors, then every basis of that vector space contains exactly n vectors. We call this number n the dimension of the vector space.

Proof

Suppose that V has one basis, S, containing exactly n vectors. If a set contains more than n vectors, that set is linearly dependent and, as such, cannot form a basis. If V had a basis with fewer than vectors, then S would be linearly dependent contradicting S being a basis. QED

A plethora of theorems about *n*-dimensional vector spaces

1. Any set of n linearly independent vectors in the space forms a basis for the space.
2. Any set of n vectors that spans the space forms a basis for the space.
3. Any set of linearly independent vectors is a subset of a basis for the space.
4. Any set of vectors that spans the space contains a subset that forms a basis for the space.

Example of theorem 3

Show that $\left\{[1,2,3]^{\mathsf{T}}, [4,5,6]^{\mathsf{T}}\right\}$ is a subset of a basis of $\mathbb{R}^3$.

$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$ is linearly independent (the vectors are not multiples) but cannot possibly span $\mathbb{R}^3$ (because bases in $\mathbb{R}^3$ have three vectors). Suppose that $\vec{v}_3$ is not in the span of the set. i.e. there are no solutions to
$$x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix} = \vec{v}_3$$

So, $\forall\, k \neq 0$, $kx_1\begin{bmatrix}1\\2\\3\end{bmatrix} + kx_2\begin{bmatrix}4\\5\\6\end{bmatrix} \neq k\vec{v}_3$

So, $\forall\, k \neq 0$ $kx_1\begin{bmatrix}1\\2\\3\end{bmatrix} + kx_2\begin{bmatrix}4\\5\\6\end{bmatrix} - k\vec{v}_3 \neq \vec{0}$

Finding bases for the null space and column space of a matrix

- A spanning set of the solution set to the homogenous system $A\vec{x}=\vec{0}$ forms a basis for the null space of A.
- The pivot columns of A form a basis for the column space of A.

Example: Let $A=\begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}$. Then $\mathrm{RREF}(A)=\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\leftarrow x_4=0$

Find a basis for the null space of A and explicitly show that this is indeed a basis for the null space of A.

The general solution to $A\vec{x}=\vec{0}$ is:

$$\begin{cases} x_1=-2x_3 \\ x_2=x_3 \\ x_3 \text{ is free} \\ x_4=0 \end{cases}$$

The null space vectors can all be written as $x_3\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}$, so $\left\{\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}\right\}$ clearly spans nul(A). Any set containing one non-zero vector has to be linearly independent.

$\therefore \left\{\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}\right\}$ forms a basis for nul(A).

check

$$\begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix}\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \checkmark$$

PIVOT COLUMNS

Example: Let $\mathbf{A} = \begin{array}{c} \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & 3 & 1 \\ -2 & 1 & -5 & 3 \\ 3 & 2 & 4 & 2 \end{bmatrix} \\ \begin{array}{cccc} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \vec{c}_4 \end{array} \end{array}$. Then $\text{RREF}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for the column space of A and explicitly show that this is indeed a basis for the column space of A.

A basis for the column space is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}.$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 1 & 0 \\ -2 & 1 & 3 & 0 \\ 3 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so}$$

the only solution to $x_1 \vec{c}_1 + x_2 \vec{c}_2 + x_3 \vec{c}_4 = \vec{0}$ is the trivial solution. Thus $\{\vec{c}_1, \vec{c}_2, \vec{c}_4\}$ is linearly independent.

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 1 & 3 \\ -2 & 1 & 3 & -5 \\ 3 & 2 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \begin{bmatrix} 3 \\ 3 \\ -5 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

So $\{\vec{c}_1, \vec{c}_2, \vec{c}_4\}$ spans $\text{col}(A)$

QED

Example

Show that any two vectors from the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 13 \\ -11 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -19 \\ 23 \end{bmatrix} \right\}$ form a basis for the span of the set.

Define

$$A = \begin{bmatrix} 2 & 3 & 0 & 11 & 9 \\ 3 & -2 & 13 & 10 & -19 \\ -1 & 4 & -11 & 0 & 23 \end{bmatrix}.$$

Then the span of the given set is $\operatorname{col}(A)$.

$$\begin{bmatrix} 2 & 3 & 0 & 11 & 9 \\ 3 & -2 & 13 & 10 & -19 \\ -1 & 4 & -11 & 0 & 23 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 4 & -3 \\ 0 & 1 & -2 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Ergo, a basis for the span of the set is $\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \right\}$.

Therefore, any two linearly independent vectors in the set form a basis for the span of the set. Since no two vectors in the set are multiples of one another, every pair of vectors in the set are linearly independent.

QED

Example

Consider the set of vectors of form $\begin{bmatrix} s \\ 2s \\ t \end{bmatrix}$; call the set H.

1. Show that H is a subspace of $\mathbb{R}^3$.
2. Every vector is H is in $\textbf{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$. Specifically, $\begin{bmatrix} s \\ 2s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Despite this fact, there is no way that $\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ forms a basis for H. Give two distinct and specific reasons why this is the case.
3. Find a basis for H.