A whole lot of equivalent properties

If A is an n x n matrix, then either each of the following statements is true about A or each of the following statements is false about A.

a. A is an invertible matrix (i.e., A is ***nonsingular***).

b. A is row equivalent to I_n.

c. A has n pivot columns.

d. The only solution to $A\vec{x}=\vec{0}$ is $\vec{0}$ (***the trivial solution***).

e. The columns of A form a linearly independent set.

f. The linear transformation $T(\vec{x})=A\vec{x}$ is one-to-one.

g. The equation $A\vec{x}=\vec{b}$ has exactly one solution $\forall\,\vec{b}\in\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $T(\vec{x})=A\vec{x}$ is onto $\mathbb{R}^n$.

l. A^{T} is nonsingular.

m. $\det(A)\neq 0$

A fact about inverse matrices

While discussing many algebraic structures, the idea of "left inverses" and "right inverses" comes up. For example, we say that b is a left inverse of a if ba equals the identity element for the structure. In fact, in more advance linear algebra classes we talk about left and right inverse matrices of non-square matrices.

We don't make that distinction for square matrices because either a square matrix has no inverse of any kind (we call such matrices ***singular***) or the left and right inverse matrices are in fact one and the same matrix. That is:

If A is a *square* matrix, then the only way that either $AB=I$ or $BA=I$ is if $B=A^{-1}$.

We're going to start with a couple of background theorems. Let's prove each of the following.

1. Prove that if T is a linear transformation, then $T\left(\vec{0}\right)=\vec{0}$.

b. Prove that the linear transformation T is one-to-one if and only if the only solution to $T\left(\vec{x}\right)=\vec{0}$ is $\vec{0}$. **Hint**: Prove the ***contrapositive statement***.

Find all values of λ that create non-trivial solutions to the system $A\vec{x} = \vec{0}$ where A is the matrix given below. Make sure that you show all relevant work and that both your reasoning and your conclusion are clear.

$$A = \begin{bmatrix} 1 & \lambda & 0 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$