A whole lot of equivalent properties

If A is an n x n matrix, then either each of the following statements is true about A or each of the following statements is false about A.

a. A is an invertible matrix (i.e., A is ***nonsingular***).

b. A is row equivalent to I_n.

c. A has n pivot columns.

d. The only solution to $A\vec{x} = \vec{0}$ is $\vec{0}$ (***the trivial solution***).

e. The columns of A form a linearly independent set.

f. The linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one.

g. The equation $A\vec{x} = \vec{b}$ has exactly one solution $\forall\, \vec{b} \in \mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $T(\vec{x}) = A\vec{x}$ is onto $\mathbb{R}^n$.

l. A^{T} is nonsingular.

m. $\det(A) \neq 0$

A fact about inverse matrices

While discussing many algebraic structures, the idea of "left inverses" and "right inverses" comes up. For example, we say that b is a left inverse of a if ba equals the identity element for the structure. In fact, in more advance linear algebra classes we talk about left and right inverse matrices of non-square matrices.

We don't make that distinction for square matrices because either a square matrix has no inverse of any kind (we call such matrices ***singular***) or the left and right inverse matrices are in fact one and the same matrix. That is:

If A is a *square* matrix, then the only way that either $AB = I$ or $BA = I$ is if $B = A^{-1}$.

$A = B$ and $B = C \implies A = C$

Transitive property of equality.

We're going to start with a couple of background theorems. Let's prove each of the following.

1. Prove that if T is a linear transformation, then $T(\vec{0}) = \vec{0}$.

b. Prove that the linear transformation T is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{0}$. **Hint:** Prove the ***contrapositive statement***.

(1) Assume that T is a linear transformation. Then, by definition,

$T(\vec{u} + \vec{0}) = T(\vec{u}) + T(\vec{0}) \quad \forall\, \vec{u} \in \text{domain}(T)$.

But $\vec{u} + \vec{0} = \vec{u}$, so $T(\vec{u} + \vec{0})$ also equals $T(\vec{u})$.

By the transitive property of equality, it follows that $T(\vec{u}) = T(\vec{u}) + T(\vec{0})$. Thus we have,

$$T \text{ a Linear Transformation} \implies T(\vec{u}) = T(\vec{u}) + T(\vec{0})$$

$$\implies T(\vec{u}) - T(\vec{u}) = T(\vec{u}) + T(\vec{0}) - T(\vec{u})$$

☆ Vector addition is commutative and associative

$$☆ \implies \vec{0} = \vec{0} + T(\vec{0})$$

$$\implies \vec{0} = T(\vec{0}) \quad \text{QED}$$

The contrapositive of $A \iff B$ is $\sim A \iff \sim B$

So we're going to prove that T is not one-to-one if and only $T(\vec{w}) = \vec{0}$ for some $\vec{w} \neq \vec{0}$.

T is not one-to-one $\iff \exists\, \vec{u}, \vec{v} \in \text{dom}(T) \ni T(\vec{u}) = T(\vec{v})$ where $\vec{u} \neq \vec{v}$.

$\iff \exists\, \vec{u}, \vec{v} \in \text{dom}(T) \ni T(\vec{u}) - T(\vec{v}) = \vec{0}$ where $\vec{u} \neq \vec{v}$

$\iff \exists$ distinct $\vec{u}, \vec{v} \in \text{dom}(T) \ni T(\vec{u} - \vec{v}) = \vec{0}$

QED

Find all values of λ that create non-trivial solutions to the system $A\vec{x} = \vec{0}$ where A is the matrix given below. Make sure that you show all relevant work and that both your reasoning and your conclusion are clear.

$$A = \begin{bmatrix} 1 & \lambda & 0 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$A\vec{x} = \vec{0}$ has nontrivial solutions iff $\det(A) = 0$ (antiproperties of det from Theorem 8)

Summing across the first row we have:

$$\det(A) = (1)(-1)^{1+1}\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + (\lambda)(-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + (0)(-1)^{1+3}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= (3-1) - \lambda(1-2)$$

$$= 2 + \lambda$$

$\therefore$ $A\vec{x} = \vec{0}$ has nontrivial solutions iff $\lambda = -2$.

Check: $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2/5 & 0 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$