Use Cramer's Rule to find the solutions to the system $\begin{cases} 3x_1 - x_2 = -10 \\ -2x_1 + 5x_2 = -2 \end{cases}$.

Swapped out x_1 coefficients for the constants

b

$$X_1 = \frac{\begin{vmatrix} -10 & -1 \\ -2 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ -2 & 5 \end{vmatrix}}$$

$$= \frac{-50-2}{15-2}$$

$$= \frac{-52}{13}$$

$$= -4$$

Swap out x_2 coefficients

$$X_2 = \frac{\begin{vmatrix} 3 & -10 \\ -2 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ -2 & 5 \end{vmatrix}}$$

$$= \frac{-6-20}{15-2}$$

$$= \frac{-26}{13}$$

$$= -2$$

The solution is $(-4, -2)$.

An algorithm for finding inverse matrices

The matrix of cofactors of a square matrix A is the matrix that results from replacing each of its entries by their corresponding cofactors.

<u>The Adjoint (Adjugate) of A</u> is the transpose of A's matrix of cofactors.

The inverse of a nonsingular square matrix A is $A^{-1} = \dfrac{1}{\det(A)}\mathrm{Adj}(A)$.

Please note that this implies that the matrix A is nonsingular if and only if $\det(A) \neq 0$. This also implies, albeit less directly, that the square system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\det(A) \neq 0$.

SQUARE MATRICES

Singular means non-invertible

nonsingular means invertible

Let's use the determinant and adjoint to find A^{-1} where $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$.

$C_{11} = (-1)^2 \begin{vmatrix} 2 & 4 \\ 3 & -3 \end{vmatrix} = -18$ $\quad C_{12} = (-1)^3 \begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} = 10$ $\quad C_{13} = (-1)^4 \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4$

$C_{21} = (-1)^3 \begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} = 3$ $\quad C_{22} = (-1)^4 \begin{vmatrix} 1 & -1 \\ 1 & -3 \end{vmatrix} = -2$ $\quad C_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -1$

$C_{31} = (-1)^4 \begin{vmatrix} 2 & -1 \\ 2 & 4 \end{vmatrix} = 10$ $\quad C_{32} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = -6$ $\quad C_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2$

$$\mathrm{Adj}(A) = \begin{bmatrix} -18 & 10 & 4 \\ 3 & -2 & -1 \\ 10 & -6 & -2 \end{bmatrix}^T$$

$$\det(A) = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{vmatrix} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= 1(-1)^2 \begin{vmatrix} 2 & 4 \\ 3 & -3 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} + 1(-1)^4 \begin{vmatrix} 2 & -1 \\ 2 & 4 \end{vmatrix}$$

$$= 1(-18) + 2(3) + 1(10)$$

$$= -2$$

$$\therefore \quad A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A) = \frac{1}{-2} \begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$