$$A \iff B \quad \text{Contrapositives} \quad \text{not } A \iff \text{not } B$$

Contrapositives are logically equivalent

We're going to start with a couple of background theorems. Let's prove each of the following.

1. Prove that if T is a linear transformation, then $T(\vec{0}) = \vec{0}$.

b. Prove that the linear transformation T is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{0}$. **Hint:** Prove the ***contrapositive statement***.

Proof of 1:

$$T(\vec{u} + \vec{0}) = T(\vec{u}) + T(\vec{0}) \quad [\text{Defining property of } T \text{ (linear transformation)}]$$

$$T(\vec{u}) = T(\vec{u}) + T(\vec{0}) \quad [\vec{u} + \vec{0} = \vec{u}]$$

$$T(\vec{u}) - T(\vec{u}) = T(\vec{u}) + T(\vec{0}) - T(\vec{u})$$

$$\vec{0} = T(\vec{0})$$

Re Proof of 1:

$$T(0 \cdot \vec{u}) = 0 \cdot T(\vec{u}) \quad [T(k\vec{u}) = kT(\vec{u})] \text{ defining property of lin. trans.}$$

$$T(\vec{0}) = \vec{0}$$

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b. The contrapositive of what we're trying to prove is:

T is not one-to-one iff $T(\vec{x}) = \vec{0}$ has nontrivial solution.

$$T \text{ is not one-to-one} \iff \exists\, \vec{u}, \vec{v};\ \vec{u} \neq \vec{v} \ni T(\vec{u}) = T(\vec{v})$$

$$\iff \exists\, \vec{u}, \vec{v};\ \vec{u} \neq \vec{v} \ni T(\vec{u}) - T(\vec{v}) = \vec{0}$$

$$\iff \exists\, \vec{u}, \vec{v};\ \vec{u} \neq \vec{v} \ni T(\vec{u} - \vec{v}) = \vec{0}$$

QED

$\vec{u} \neq \vec{v}$ so $\vec{u} - \vec{v} \neq \vec{0}$

$A\vec{x} = \vec{b}$

Find all values of λ that create non-trivial solutions to the system $A\vec{x} = \vec{0}$ where A is the matrix given below. Make sure that you show all relevant work and that both your reasoning and your conclusion are clear.

$$A = \begin{bmatrix} 1 & \lambda & 0 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$A\vec{x} = \vec{0}$ has non-trivial solutions iff $\det(A) = 0$

[This is the contrapositive of $d \Leftrightarrow m$ from page 10 of the determinant notes].

$$\det(A) = \begin{vmatrix} 1 & \lambda & 0 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= (1)(-1)^{1+1}\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + \lambda(-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 0(-1)^{1+3}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= 2 + \lambda$$

$\det(A) = 0$ iff $\lambda = -2$

$\therefore$ $A\vec{x} = \vec{0}$ has non-trivial solutions iff $\lambda = -2$.