Let $T(\vec{0}) = \vec{w}$. Then, $\forall \vec{u} \in \text{range}(T)$

$$T(\vec{u}) = T(\vec{u}+\vec{0})$$
$$= T(\vec{u}) + T(\vec{0})$$

$$T(\vec{u}) = T(\vec{u}) + T(\vec{0})$$
$$\therefore \quad T(\vec{0}) = T(\vec{u}) - T(\vec{u})$$
$$= \vec{0}$$

1b. <u>contrapositive statement</u>

T is <u>not</u> one-to-one iff $T(\vec{x}) = \vec{0}$ <u>has</u> <u>non</u> trivial solutions.

$$T \text{ is not one-to-one} \iff \exists\, \vec{u} \neq \vec{v} \ni T(\vec{u}) = T(\vec{v})$$
$$\iff T(\vec{u}) - T(\vec{v}) = \vec{0}, \vec{u} \neq \vec{v}$$
$$\iff T(\vec{u}-\vec{v}) = \vec{0}, \vec{u}-\vec{v} \neq \vec{0}$$

— QED

2a: Properties "b" and "c" are identical

Pivot positions precisely correspond to leading entries of the reduced echelon form of the matrix.

Ergo, $A \sim I_n \iff A$ has n pivot positions.

b. "g", "h" and "i" are three ways of saying "g"

"h": The columns of a span $\mathbb{R}^n$

$$\underbrace{x_1 \vec{C}_1 + x_2 \vec{C}_2 + \dots x_n \vec{C}_n = \vec{b}}_{A\vec{x} = \vec{b}}$$ always has a solution

"i" $\underbrace{T(\vec{x}) = A\vec{x} \text{ is onto } \mathbb{R}^n}$

"g" $\longrightarrow T(\vec{x}) = A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^n$

C. Why is the # of pivot positions determinative of whether a matrix is singular (no inverse) or non-singular (inverse)

of

lemma

A matrix with even one row of zeros is singular.

Proof: Suppose that the k^{th} row of A is all zeros. Then, the $\underline{k^{th} \text{ row}}$ of AB is $\underline{\text{all zeros}}$ $\forall B$ QED*

*(no way does $AB = I$, cuz I don't got no row of zeros)

"a" When we execute the algorithm $[A \vdots I] \sim [I \vdots A^{-1}]$ what we're really doing is solving n systems of equations simultaneously. eg. If $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, when we write $\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right]$, we're actually simultaneously solving $M\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $M\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $M\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

If A has n pivot-positions, $A \sim I$ and all three of those equations have solutions.
If A doesn't have n pivot-positions, In echelon form, A has a row of zeros.

$$A_{3\times3} \quad [A \vdots I] \sim \left[\begin{array}{ccc|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & a & b & c \end{array}\right]$$

a, b + c cannot all be zero, because this would imply A^{-1} has a row of zeros (see lemma). Since one of a, b, c must be nonzero, one of our equations is a contradiction

$$0+0+0=a$$
$$0+0+0=b$$
$$0+0+0=c$$

"⇐" $A_{n \times n}$ does not have n pivot positions

$\iff$ $A \sim B$, where B is a reduced echelon form matrix with a row of zeros

$\iff$ $\det(B) = 0$ (echelon form matrices are upper triangular matrices, so the determinant value is the product of the main diagonal entries.)

and since $\det(A) = k \det(B)$,

$\det(A) = 0$