Determinants (of square matrices)

$$\det([a_{11}]) = |\,a_{11}\,| = a_{11}$$

For a square matrix, A, with two or more rows we define the cofactor of entry $\underline{\boldsymbol{a_{ij}}}$, $\boldsymbol{C_{ij}}$, to be $(-1)^{i+j}$ times the determinant of the matrix that results from eliminating the i^{th} row and j^{th} column from A. Then using any row of A or any column of A:

$$\det(\mathrm{A}) = \sum_{j=1}^{n}\left[a_{ij}\, C_{ij}\right] = \sum_{i=1}^{n}\left[a_{ij}\, C_{ij}\right]$$

Please note that in the first formula we are summing along a fixed i^{th} row of A whereas in the second formula we are summing along a fixed j^{th} column of A.

Example

Find a simplified formula for $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ - first by summing along the first row and again by summing along the second column.

Example

Use cofactors along the second row to find $\det(A)$ where $A = \begin{bmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{bmatrix}$. Verify the determinant value by using cofactors along the first column.

Example: Evaluate the determinate; the equal sign has been introduced to save space.

$$\begin{vmatrix} 3 & 9 & 0 & -1 \\ 0 & -3 & -2 & 7 \\ 2 & 5 & 0 & 4 \\ 0 & -6 & 0 & 6 \end{vmatrix} =$$

Example

Use a determinant to find $\vec{u} \times \vec{v}$ where $\vec{u} = [1, 7, -3]$ and $\vec{v} = [3, 0, 5]$.

Elementary Matrices

An ***elementary matrix*** is a matrix that can be created from an identity matrix via one elementary row operation.

Example: Let $B=\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$.

For each of the following matrices (each identified as A), describe the row operation that was affected upon I_2 to create A. Then find $\det(A)$ and compare its value to $\det(I_2)$. Next, find AB and describe the difference between it and B. Finally, compare the values of $\det(AB)$ and $\det(B)$. This example continues on page 5.

$$A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$A=\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

Elementary Row Operations and Determinants

Suppose that A and B are square matrices of equal dimension; suppose further that B can be created from A via a single elementary row operation. Then:

- if the operation is adding a multiple of one row of A to a different row of A, then $\det(B) = \det(A)$
- if the operation is swapping two rows of A, then $\det(B) = -\det(A)$
- if the operation is multiplying a row of A by the real number k, then $\det(B) = k \cdot \det(A)$.

A couple of definitions and a convenient fact

An ***upper triangular matrix*** is a matrix where every entry *below* the main diagonal is zero.

A ***lower triangular matrix*** is a matrix where every entry *above* the main diagonal is zero.

The determinant of any $n \times n$ triangular matrix B is given by the formula $\det(B) = \prod_{i=1}^{n} b_{ii}$.

Example

Determine $\begin{vmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{vmatrix}$ after first manipulating the matrix into upper triangular form.

Determine $\det(A)$ where $A=\begin{bmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 9 & 2 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{bmatrix}$ after first manipulating the matrix into upper triangular form.

A couple of things that are true about determinants

If A and B are like-sized square matrices, then $\det(AB) = \det(A)\det(B)$.

If A is any square matrix, then $\det(A^{\mathrm{T}}) = \det(A)$.

Example

Prove the first thing is true for two by two matrices using $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

A Little Geometry

$\left|\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right|$ finds the area of any parallelogram whose sides are parallel to $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

Example

Find the area of the parallelogram outline in Figure 1.

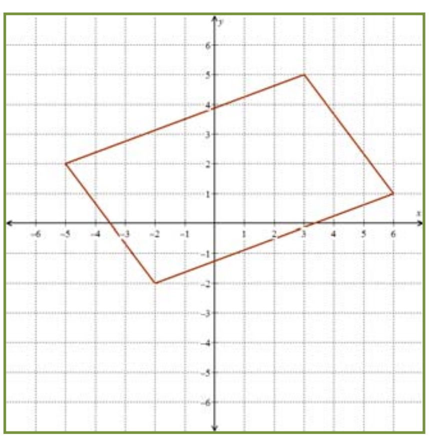

Figure 1: A parallelogram

Use Cramer's Rule to find the solutions to the system $\begin{cases} 3x_1 - x_2 = -10 \\ -2x_1 + 5x_2 = -2 \end{cases}$.

An algorithm for finding inverse matrices

The matrix of cofactors of a square matrix A is the matrix that results from replacing each of its entries by their corresponding cofactors.

The Adjoint (Adjugate) of A is the transpose of A's matrix of cofactors.

The inverse of a nonsingular square matrix A is $A^{-1} = \frac{1}{\det(A)}\text{Adj}(A)$.

Please note that this implies that the matrix A is nonsingular if and only if $\det(A) \neq 0$. This also implies, albeit less directly, that the square system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\det(A) \neq 0$.

Let's use the determinant and adjoint to find A^{-1} where $A=\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$.