Determinants (of square matrices)

$$\det([a_{11}]) = |a_{11}| = a_{11}$$

For a square matrix, A, with two or more rows we define the cofactor of entry $\underline{a_{ij}}$, C_{ij}, to be $(-1)^{i+j}$ times the determinant of the matrix that results from eliminating the i^{th} row and j^{th} column from A. Then using any row of A or any column of A:

$$\det(A) = \sum_{j=1}^{n}\left[a_{ij}\,C_{ij}\right] = \sum_{i=1}^{n}\left[a_{ij}\,C_{ij}\right]$$

Please note that in the first formula we are summing along a fixed i^{th} row of A whereas in the second formula we are summing along a fixed j^{th} column of A.

Example

Find a simplified formula for $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ - first by summing along the first row and again by summing along the second column.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sum_{j=1}^{2}\left[a_{1,j}\,C_{1,j}\right]$$
$$= a_{11}C_{11} + a_{12}C_{12}$$
$$= a_{11}\cdot(-1)^{1+1}\cdot|a_{22}| + a_{12}\cdot(-1)^{1+2}\cdot|a_{21}|$$
$$= a_{11}\cdot 1\cdot a_{22} + a_{12}\cdot(-1)\cdot a_{21}$$
$$= a_{11}a_{22} - a_{12}a_{21}$$

Caution

In this context, $|a_{22}|$ we're talking determinants, not absolute value. $|x| = x$. If $A = [x]$, $\det(A) = x$.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sum_{i=1}^{2}\left[a_{i,2}\,C_{i,2}\right]$$
$$= a_{12}C_{12} + a_{22}C_{22}$$
$$= a_{12}\cdot(-1)^{1+2}\cdot|a_{21}| + a_{22}\cdot(-1)^{2+2}|a_{11}|$$
$$= a_{12}\cdot(-1)\cdot a_{21} + a_{22}\cdot 1\cdot a_{11}$$
$$= a_{11}a_{22} - a_{12}a_{21}$$

Example

Use cofactors along the second row to find $\det(A)$ where $A=\begin{bmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{bmatrix}$. Verify the determinant value by using cofactors along the first column.

Across 2^{nd} Row ...

$$\det(A) = \sum_{j=1}^{3} \left[a_{2,j} C_{2,j} \right]$$

$$= a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}$$

$$= (3)\cdot(-1)^{2+1} \cdot \begin{vmatrix} 6 & -1 \\ 9 & 3 \end{vmatrix} + (1)\cdot(-1)^{2+2} \cdot \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + (-4)\cdot(-1)^{2+3} \cdot \begin{vmatrix} 2 & 6 \\ 1 & 9 \end{vmatrix}$$

$$= (3)(-1)\cdot(18-(-9)) + (1)(1)(6-(-1)) + (-4)(-1)(18-6)$$

$$= -81 + 7 + 48$$

$$= -26 \;\#$$

Summing along the 1^{st} Column

$$\begin{vmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{vmatrix} = \sum_{i=1}^{3} \left[a_{i,1} C_{i,1} \right]$$

$$= a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31}$$

$$= (2)(-1)^{1+1} \begin{vmatrix} 1 & -4 \\ 9 & 3 \end{vmatrix} + (3)(-1)^{2+1} \begin{vmatrix} 6 & -1 \\ 9 & 3 \end{vmatrix} + (1)(-1)^{3+1} \begin{vmatrix} 6 & -1 \\ 1 & -4 \end{vmatrix}$$

$$= 2(3-(-36)) - 3(18-(-9)) + 1\cdot(-24-(-1))$$

$$= 78 - 81 - 23$$

$$= -26 \;\#$$

It's craziness to not sum down the third column

Example: Evaluate the determinate; the equal sign has been introduced to save space.

$$\begin{vmatrix} 3 & 9 & 0 & -1 \\ 0 & -3 & -2 & 7 \\ 2 & 5 & 0 & 4 \\ 0 & -6 & 0 & 6 \end{vmatrix} = \sum_{i=1}^{4} [a_{i,3} C_{i,3}]$$

$$= a_{13} C_{13} + a_{23} C_{23} + a_{33} C_{33} + a_{43} C_{43}$$

$$= (0) \cdot (-1)^{1+3} \cdot \begin{vmatrix} 0 & -3 & 7 \\ 2 & 5 & 4 \\ 0 & -6 & 6 \end{vmatrix} + (-2) \cdot (-1)^{2+3} \cdot \begin{vmatrix} 3 & 9 & -1 \\ 2 & 5 & 4 \\ 0 & -6 & 6 \end{vmatrix}$$

$$+ (0) \cdot (-1)^{3+3} \cdot \begin{vmatrix} 3 & 9 & -1 \\ 0 & -3 & 7 \\ 0 & -6 & 6 \end{vmatrix} + (0) \cdot (-1)^{4+3} \cdot \begin{vmatrix} 3 & 9 & -1 \\ 0 & -3 & 7 \\ 2 & 5 & 4 \end{vmatrix}$$

$$= (-2)(-1) \cdot \begin{vmatrix} 3 & 9 & -1 \\ 2 & 5 & 4 \\ 0 & -6 & 6 \end{vmatrix} \leftarrow \text{Sum along } 1^{st} \text{ row}$$

$$= 2 \cdot \left[(3)(-1)^{1+1} \cdot \begin{vmatrix} 5 & 4 \\ -6 & 6 \end{vmatrix} + (9)(-1)^{1+2} \cdot \begin{vmatrix} 2 & 4 \\ 0 & 6 \end{vmatrix} + (-1)(-1)^{1+3} \cdot \begin{vmatrix} 2 & 5 \\ 0 & -6 \end{vmatrix} \right)$$

$$= 2 \left[3(30 - (-24)) - 9(12 - 0) + (-1)(-12 - 0) \right]$$

$$= 2 \left(162 - 108 + 12 \right]$$

$$= 132 \checkmark$$

Example

Use a determinant to find $\vec{u} \times \vec{v}$ where $\vec{u} = [1, 7, -3]$ and $\vec{v} = [3, 0, 5]$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 7 & -3 \\ 3 & 0 & 5 \end{vmatrix}$$

$$= \sum_{j=1}^{3} [a_{1,j} C_{1,j}]$$

$$= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= \hat{i} \cdot (-1)^{1+1} \cdot \begin{vmatrix} 7 & -3 \\ 0 & 5 \end{vmatrix} + \hat{j} \cdot (-1)^{1+2} \cdot \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} + \hat{k} \cdot (-1)^{1+3} \cdot \begin{vmatrix} 1 & 7 \\ 3 & 0 \end{vmatrix}$$

$$= \hat{i} \cdot (35 - 0) - \hat{j} \cdot (5 - (-9)) + \hat{k} \cdot (0 - 21)$$

$$= 35\hat{i} - 14\hat{j} - 21\hat{k}$$

Check

$$(\vec{u} \times \vec{v}) \cdot \vec{u} = 35 - 98 + 63$$
$$= 0 \checkmark$$

$$(\vec{u} \times \vec{v}) \cdot \vec{v} = 105 + 0 - 105$$
$$= 0 \checkmark$$

Elementary Matrices

An ***elementary matrix*** is a matrix that can be created from an identity matrix via one elementary row operation.

Example: Let $B=\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$.

For each of the following matrices (each identified as A), describe the row operation that was affected upon I_2 to create A. Then find $\det(A)$ and compare its value to $\det(I_2)$. Next, find AB and describe the difference between it and B. Finally, compare the values of $\det(AB)$ and $\det(B)$. This example continues on page 5.

$A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ $\quad I \xrightarrow{R_1 \leftrightarrow R_2} A$ (row swap)

$\det(A) = (0)(0) - (1)(1)$
$= -1$
$= -\det(I)$ (Sign change on det value)

$\boxed{\det(B) = (3)(1) - (-1)(2) = 5}$

$AB = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$
$= \begin{bmatrix}0+2 & 0+1\\ 3+0 & -1+0\end{bmatrix}$
$= \begin{bmatrix}2 & 1\\ 3 & -1\end{bmatrix}$ Same row swap ($B \xrightarrow{R_1 \leftrightarrow R_2} AB$)

$\det(AB) = (2)(-1) - (1)(3)$
$= -5$
$= -\det(B)$
Sign change on det value

$A=\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}$ $\quad I \xrightarrow{3R_1 \to R_1} A$

$\det(A) = (3)(1) - (0)(0)$
$= 3$
$= 3\cdot\det(I)$ ← The determinate value of I increased by a factor of 3.

$AB = \begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$
$= \begin{bmatrix}(3)(3)+(0)(2) & (3)(-1)+(0)(1)\\ (0)(3)+(1)(2) & (0)(-1)+(1)(1)\end{bmatrix}$
$= \begin{bmatrix}9 & -3\\ 2 & 1\end{bmatrix}$ $\quad B \underbrace{\xrightarrow{3R_1 \to R_1}}_{\text{Same operation as above}} AB$

$\det(AB) = (9)(1) - (-3)(2)$
$= 15$
$= 3\cdot 5$
$= 3\det(B)$

$$A=\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$$

$I \quad -2R_2 \to R_2 \quad A$

$\det(A) = (1)(-2) - (0)(0)$
$= -2$
$= -2\det(I)$

The value of the determinant of I was multiplied by -2

$AB = \begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$
$= \begin{bmatrix}(1)(3)+(0)(2) & (1)(-1)+(0)(1)\\ (0)(3)+(-2)(2) & (0)(-1)+(-2)(1)\end{bmatrix}$
$= \begin{bmatrix}3 & -1\\ -4 & -2\end{bmatrix}$

$B \quad -2R_2 \to R_2 \quad AB$

$\det(AB) = (3)(-2) - (-1)(-4)$
$= -10$
$= -2\det(B)$
!!

$$A=\begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}$$

$I \quad -3R_1 + R_2 \to R_2 \quad A$

$\det(A) = (1)(1) - (0)(-3)$
$= (1)$
$= \det(I)$

The determinant value did not change.

$AB = \begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$

Gee willikers

$= \begin{bmatrix}(1)(3)+(0)(2) & (1)(-1)+(0)(1)\\ (-3)(3)+(1)(2) & (-3)(-1)+(1)(1)\end{bmatrix}$

$B \quad -3R_1 + R_2 \to R_2 \quad AB$

$= \begin{bmatrix}3 & -1\\ -7 & 4\end{bmatrix}$

$\det(AB) = (3)(4) - (-1)(-7)$
$= 5$
$= \det(B)$ ✓

$$A=\begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}$$

$I \quad 4R_2 + R_1 \to R_1 \quad A$

$\det(A) = (1)(1) - (4)(0)$
$= 1$
$= \det(I)$

no change in determinant value

$AB = \begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$

$\det(AB) = (11)(1) - (3)(2)$
$= 5$
$= \det(B)$ ✓

$= \begin{bmatrix}(1)(3)+(4)(2) & (1)(-1)+(4)(1)\\ (0)(3)+(1)(2) & (0)(-1)+(1)(1)\end{bmatrix}$
$= \begin{bmatrix}11 & 3\\ 2 & 1\end{bmatrix}$

$B \quad 4R_2 + R_1 \to R_1 \quad AB$

Elementary Row Operations and Determinants

Suppose that A and B are square matrices of equal dimension; suppose further that B can be created from A via a single elementary row operation. Then:

- if the operation is adding a multiple of one row of A to a different row of A, then $\det(B) = \det(A)$
- if the operation is swapping two rows of A, then $\det(B) = -\det(A)$
- if the operation is multiplying a row of A by the real number k, then $\det(B) = k \cdot \det(A)$.

A couple of definitions and a convenient fact

An ***upper triangular matrix*** is a matrix where every entry *below* the main diagonal is zero.

A ***lower triangular matrix*** is a matrix where every entry *above* the main diagonal is zero.

The determinant of any $n \times n$ triangular matrix B is given by the formula $\det(B) = \prod_{i=1}^{n} b_{ii}$.

Example

Determine $\begin{vmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{vmatrix}$ after first manipulating the matrix into upper triangular form.

Effect on Determinant

$$\begin{vmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{vmatrix} \quad R_1 \leftrightarrow R_3 \quad \begin{vmatrix} 1 & 9 & 3 \\ 3 & 1 & -4 \\ 2 & 6 & -1 \end{vmatrix}$$

Sign swap

$$\begin{matrix} -3R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{matrix} \quad \begin{vmatrix} 1 & 9 & 3 \\ 0 & -26 & -13 \\ 0 & -12 & -7 \end{vmatrix}$$

no change

$$-\frac{12}{26}R_2 + R_3 \to R_3 \quad \begin{vmatrix} 1 & 9 & 3 \\ 0 & -26 & -13 \\ 0 & 0 & -1 \end{vmatrix}$$

nochange

$$\therefore \begin{vmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 9 & 3 \\ 0 & -26 & -13 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= -(1)(-26)(-1)$$

$$= -26 \#$$

Determine $\det(A)$ where $A = \begin{bmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 9 & 2 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{bmatrix}$ after first manipulating the matrix into upper triangular form.

effect on det

$$\begin{vmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 9 & 2 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{vmatrix} \quad \begin{matrix} R_1 \leftrightarrow R_2 \\ R_3 \leftrightarrow R_4 \end{matrix} \quad \begin{vmatrix} 1 & 0 & 9 & 2 \\ 3 & 1 & -1 & 4 \\ 4 & 2 & 6 & -1 \\ 8 & -3 & 1 & 7 \end{vmatrix}$$

$(-)(-)$ no change.

$$\begin{matrix} -3R_1 + R_2 \to R_2 \\ -4R_1 + R_3 \to R_3 \\ -8R_1 + R_4 \to R_4 \end{matrix} \quad \begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 2 & -30 & -9 \\ 0 & -3 & -71 & -9 \end{vmatrix}$$

no change

$$\begin{matrix} -2R_2 + R_3 \to R_3 \\ 3R_2 + R_4 \to R_4 \end{matrix} \quad \begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 0 & 26 & -5 \\ 0 & 0 & -155 & -15 \end{vmatrix}$$

no change

$$\frac{155}{26} R_3 + R_4 \to R_4 \quad \begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 0 & 26 & -5 \\ 0 & 0 & 0 & -\frac{1165}{26} \end{vmatrix}$$

no change

$$\therefore \quad \det(A) = \begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 0 & 26 & -5 \\ 0 & 0 & 0 & -\frac{1165}{26} \end{vmatrix}$$

$$= (1)(1)(26)(-1165/26)$$

$$= -1165$$

Effect

$\begin{vmatrix} 5 & 15 \\ -3 & 2 \end{vmatrix}$ $\quad \frac{1}{5}R_1 \to R_1 \quad$ $\begin{vmatrix} 1 & 3 \\ -3 & 2 \end{vmatrix}$ $\qquad$ new det $= \frac{1}{5}$ original det.

$2R_2 \to R_2 \quad \begin{vmatrix} 1 & 3 \\ -6 & 4 \end{vmatrix}$ $\qquad$ new det $= 2$ last det.

$6R_1 + R_2 \to R_2 \quad \begin{vmatrix} 1 & 3 \\ 0 & 22 \end{vmatrix}$ $\qquad$ no change

$$\therefore \begin{vmatrix} 5 & 15 \\ -3 & 2 \end{vmatrix} = \frac{5}{2}\begin{vmatrix} 1 & 3 \\ 0 & 22 \end{vmatrix}$$

$$= \frac{5}{2}(22)$$

$$= 55 \checkmark$$

A couple of things that are true about determinants

If A and B are like-sized square matrices, then $\det(AB) = \det(A)\det(B)$.

If A is any square matrix, then $\det(A^{\mathrm{T}}) = \det(A)$.

Example

Prove the first thing is true for two by two matrices using $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

$$\det(AB) = \begin{vmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{vmatrix}$$

$$= (ax+bz)(cy+dw) - (ay+bw)(cx+dz)$$

$$= \cancel{acxy} + adwx + bcyz + \cancel{bdwz} - \cancel{acxy} - adyz - bcwx - \cancel{bdwz}$$

$$= adwx - adyz + bcyz - bcwx$$

$$= ad(wx - yz) - bc(wx - yz)$$

$$= (ad - bc)(wx - yz)$$

$$= \det(A)\det(B) \quad \text{woo hoo!}$$

A Little Geometry

$\left|\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right|$ finds the area of any parallelogram whose sides are parallel to $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

abs. value   determinant

Example

Find the area of the parallelogram outline in Figure 1.

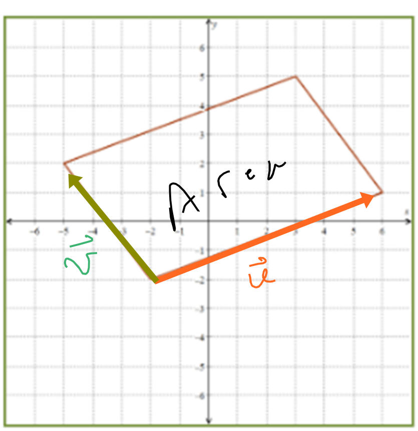

Figure 1: A parallelogram

$$\vec{u} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\text{Area} = \left|\begin{vmatrix} 8 & -3 \\ 3 & 4 \end{vmatrix}\right|$$

$$= 41$$

Use Cramer's Rule to find the solutions to the system $\begin{cases} 3x_1 - x_2 = -10 \\ -2x_1 + 5x_2 = -2 \end{cases}$.

x_1 coefficients replaced with constants

$$x_1 = \frac{\begin{vmatrix} -10 & -1 \\ -2 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ -2 & 5 \end{vmatrix}} \leftarrow \text{coefficient matrix determinant}$$

$$= \frac{-52}{13}$$

$$= -4$$

x_2 coefficients replaced with constants

$$x_2 = \frac{\begin{vmatrix} 3 & -10 \\ -2 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ -2 & 5 \end{vmatrix}}$$

$$= \frac{-26}{13}$$

$$= -2$$

$\therefore$ The solution is $(-4, -2)$

J - check ✓

An algorithm for finding inverse matrices

The matrix of cofactors of a square matrix A is the matrix that results from replacing each of its entries by their corresponding cofactors.

The Adjoint (Adjugate) of A is the transpose of A's matrix of cofactors.

The inverse of a nonsingular square matrix A is $A^{-1} = \frac{1}{\det(A)}\text{Adj}(A)$.

Please note that this implies that the matrix A is nonsingular if and only if $\det(A) \neq 0$. This also implies, albeit less directly, that the square system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\det(A) \neq 0$.

Let's use the determinant and adjoint to find A^{-1} where $A=\begin{bmatrix}1 & 2 & -1\\ 2 & 2 & 4\\ 1 & 3 & -3\end{bmatrix}$.

$C_{11}=(-1)^{1+1}\cdot\begin{vmatrix}2 & 4\\ 3 & -3\end{vmatrix}$
$=-18$

$C_{12}=(-1)^{1+2}\begin{vmatrix}2 & 4\\ 1 & -3\end{vmatrix}$
$=10$

$C_{13}=(-1)^{1+3}\begin{vmatrix}2 & 2\\ 1 & 3\end{vmatrix}$
$=4$

$C_{21}=(-1)^{2+1}\begin{vmatrix}2 & -1\\ 3 & -3\end{vmatrix}$
$=3$

$C_{22}=(-1)^{2+2}\begin{vmatrix}1 & -1\\ 1 & -3\end{vmatrix}$
$=-2$

$C_{23}=(-1)^{2+3}\begin{vmatrix}1 & 2\\ 1 & 3\end{vmatrix}$
$=-1$

$C_{31}=(-1)^{3+1}\begin{vmatrix}2 & -1\\ 2 & 4\end{vmatrix}$
$=10$

$C_{32}=(-1)^{3+2}\begin{vmatrix}1 & -1\\ 2 & 4\end{vmatrix}$
$=-6$

$C_{33}=(-1)^{3+3}\begin{vmatrix}1 & 2\\ 2 & 2\end{vmatrix}$
$=-2$

$$\therefore\quad A^{-1}=\frac{1}{\det(A)}\cdot \mathrm{Adj}(A)$$

$$=\frac{1}{-2}\begin{bmatrix}-18 & 3 & 10\\ 10 & -2 & -6\\ 4 & -1 & -2\end{bmatrix}$$

$$=\begin{bmatrix}9 & -1.5 & -5\\ -5 & 1 & 3\\ -2 & 0.5 & 1\end{bmatrix}$$

A T In Check ✓