Determinants (of square matrices)

$$\det([a_{11}]) = |\, a_{11} \,| = a_{11}$$

For a square matrix, A, with two or more rows we define the cofactor of entry $\underline{a_{ij}}$, C_{ij}, to be $(-1)^{i+j}$ times the determinant of the matrix that results from eliminating the i^{th} row and j^{th} column from A. Then using any row of A or any column of A:

$$\det(\mathrm{A}) = \sum_{j=1}^{n}\left[a_{ij}\, C_{ij}\right] = \sum_{i=1}^{n}\left[a_{ij}\, C_{ij}\right]$$

Please note that in the first formula we are summing along a fixed i^{th} row of A whereas in the second formula we are summing along a fixed j^{th} column of A.

Example

Find a simplified formula for $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ - first by summing along the first row and again by summing along the second column.

Contextually, $|a_{ij}|$ means the determinant of $[a_{ij}]$ which is the number a_{ij}

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sum_{j=1}^{2}\left[a_{1,j}\, C_{1,j}\right]$$

$$= a_{11} C_{11} + a_{12} C_{12}$$

$$= a_{11} (-1)^{1+1} |a_{22}| + a_{12} (-1)^{1+2} |a_{21}|$$

$$= a_{11} (1)\, a_{22} + a_{12} (-1)\, a_{21}$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sum_{i=1}^{2}\left[a_{i,2}\, C_{i,2}\right]$$

$$= a_{12} C_{12} + a_{22} C_{22}$$

$$= a_{12} (-1)^{1+2} |a_{21}| + a_{22} (-1)^{2+2} |a_{11}|$$

$$= a_{12} (-1)\, a_{21} + a_{22} (1)\, a_{11}$$

$$= -a_{12} a_{21} + a_{22} a_{11}$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

Example

Use cofactors along the second row to find $\det(A)$ where $A=\begin{bmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{bmatrix}$. Verify the determinant value by using cofactors along the first column.

$$\det(A) = \sum_{j=1}^{3} \left[a_{2,j} C_{2,j} \right]$$

$$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= 3(-1)^{2+1}\begin{vmatrix} 6 & -1 \\ 9 & 3 \end{vmatrix} + (1)(-1)^{2+2}\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + (-4)(-1)^{2+3}\begin{vmatrix} 2 & 6 \\ 1 & 9 \end{vmatrix}$$

$$= 3(-1)(18-(-9)) + (1)(1)(6-(-1)) + (-4)(-1)(18-6)$$

$$= -81 + 7 + 48$$

$$= -26$$

$$\det(A) = \sum_{i=1}^{3} \left[a_{i,1} C_{i,1} \right]$$

$$= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= 2(-1)^{1+1}\begin{vmatrix} 1 & -4 \\ 9 & 3 \end{vmatrix} + 3(-1)^{2+1}\begin{vmatrix} 6 & -1 \\ 9 & 3 \end{vmatrix} + (1)(-1)^{3+1}\begin{vmatrix} 6 & -1 \\ 1 & -4 \end{vmatrix}$$

$$= 2(1)(3-(-36)) + 3(-1)(18-(-9)) + (1)(1)(-24-(-1))$$

$$= 2(39) - 3(27) - 23$$

$$= 78 - 81 - 23$$

$$= -26$$

Zero-Zone! Sum down column three

Example: Evaluate the determinate; the equal sign has been introduced to save space.

$$\begin{vmatrix} 3 & 9 & 0 & -1 \\ \cancel{0} & \cancel{-3} & \cancel{2} & \cancel{7} \\ 2 & 5 & 0 & 4 \\ 0 & -6 & 0 & 6 \end{vmatrix} = \sum_{i=1}^{4} \left[a_{i,3} C_{i,3} \right]$$

$$= a_{13} C_{13} + a_{23} C_{23} + a_{33} C_{33} + a_{43} C_{43}$$

$$= (0) C_{11} + (-2)(-1)^{2+3} \begin{vmatrix} 3 & 9 & -1 \\ 2 & 5 & 4 \\ 0 & -6 & 6 \end{vmatrix} + (0) C_{33} + (0) C_{43}$$

$$= (-2)(-1) \left[a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31} \right]$$

$$= 2 \left[3(-1)^2 \begin{vmatrix} 5 & 4 \\ -6 & 6 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 9 & -1 \\ -6 & 6 \end{vmatrix} + 0(-1)^4 \begin{vmatrix} 9 & -1 \\ 5 & 4 \end{vmatrix} \right]$$

$$= 2 \left[3(54) - 2(48) + 0 \right]$$

$$= 132$$

Example

Use a determinant to find $\vec{u} \times \vec{v}$ where $\vec{u} = [1, 7, -3]$ and $\vec{v} = [3, 0, 5]$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 7 & -3 \\ 3 & 0 & 5 \end{vmatrix}$$

$$= \sum_{j=1}^{3} \left(a_{1,j} C_{1,j} \right]$$

$$= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= \hat{i} (-1)^{1+1} \begin{vmatrix} 7 & -3 \\ 0 & 5 \end{vmatrix} + \hat{j} (-1)^{1+2} \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} + \hat{k} (-1)^{1+3} \begin{vmatrix} 1 & 7 \\ 3 & 0 \end{vmatrix}$$

$$= \hat{i} (35 - (0)) - \hat{j} (5 - (-9)) + \hat{k} (0 - 21)$$

Checkable!

$\langle 35, -14, -21 \rangle \cdot \langle 1, 7, -3 \rangle = 0$

$\langle 35, -14, -21 \rangle \cdot \langle 3, 0, 5 \rangle = 0$

$$= \langle 35, -14, -21 \rangle$$

Elementary Matrices

An ***elementary matrix*** is a matrix that can be created from an identity matrix via one elementary row operation.

Example: Let $B=\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$.

For each of the following matrices (each identified as A), describe the row operation that was ~~e~~affected upon I_2 to create A. Then find $|A|$ and compare its value to $|I_2|$. Next, find AB and describe the difference between it and B. Finally, compare the values of $|AB|$ and $|B|$. This example continues on page 5.

$A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

$I_2 \xrightarrow{R_1 \leftrightarrow R_2} A$

$\det(A) = -\det(I)$

$\det(I_2) = \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} = 1-0 = 1$

$\det(A) = \begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix} = 0-1 = -1$

$AB = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix} = \begin{bmatrix}0+2 & 0+1\\ 3+0 & -1+0\end{bmatrix} = \begin{bmatrix}2 & 1\\ 3 & -1\end{bmatrix}$

AB: $B \xrightarrow{R_1 \leftrightarrow R_2} AB$ (just like $I_2 \xrightarrow{R_1 \leftrightarrow R_2} A$)

$\det(B) = \begin{vmatrix}3 & -1\\ 2 & 1\end{vmatrix} = 3-(-2) = 5$; $\det(AB) = \begin{vmatrix}2 & 1\\ 3 & -1\end{vmatrix} = -2-3 = -5 = -\det(B)$

$A=\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}$

$I \xrightarrow{3R_1 \to R_1} A$

$\det(A) = 3 = 3\cdot\det(I)$

$AB = \begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix} = \begin{bmatrix}3(3)+0(2) & 3(-1)+0(1)\\ 0(3)+(1)(2) & 0(-1)+(1)(1)\end{bmatrix} = \begin{bmatrix}9 & -3\\ 2 & 1\end{bmatrix}$

B: $3R_1 \to R_1$ AB

$\det(AB) = \begin{vmatrix}9 & -3\\ 2 & 1\end{vmatrix} = 15 = 3(5) = 3\det(B)$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

$I \quad 4R_2 + R_1 \to R_1 \quad A$

$\det(A) = \begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \quad \det(I)$

$$AB = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} (1)(3) + (4)(2) & (1)(-1) + 4(1) \\ (0)(3) + (1)(2) & (0)(-1) + (1)(1) \end{bmatrix} = \begin{bmatrix} 11 & 3 \\ 2 & 1 \end{bmatrix}$$

$\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = B \quad 4R_2 + R_1 \quad \begin{bmatrix} 11 & 3 \\ 2 & 1 \end{bmatrix} = AB$

$\det(AB) = 11 - 6 = 5 = \det(B)$

Elementary Row Operations and Determinants

Suppose that A and B are square matrices of equal dimension; suppose further that B can be created from A via a single elementary row operation. Then:

- if the operation is adding a multiple of one row of A to a different row of A, then $\det(B) = \det(A)$
- if the operation is swapping two rows of A, then $\det(B) = -\det(A)$
- if the operation is multiplying a row of A by the real number k, then $\det(B) = k \cdot \det(A)$.

A couple of definitions and a convenient fact

An ***upper triangular matrix*** is a matrix where every entry *below* the main diagonal is zero.

A ***lower triangular matrix*** is a matrix where every entry *above* the main diagonal is zero.

The determinant of any $n \times n$ triangular matrix B is given by the formula $\det(B) = \prod_{i=1}^{n} b_{ii}$.

Example

Determine $\begin{vmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{vmatrix}$ after first manipulating the matrix into upper triangular form.

Row manipulation | EFFECT

$$\begin{vmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{vmatrix} \quad R_1 \leftrightarrow R_3 \quad \begin{vmatrix} 1 & 9 & 3 \\ 3 & 1 & -4 \\ 2 & 6 & -1 \end{vmatrix}$$

(-1) sign change

$$\begin{matrix} -3R_1 + R_2 \rightarrow R_2 \\ -2R_1 + R_3 \rightarrow R_3 \end{matrix} \quad \begin{vmatrix} 1 & 9 & 3 \\ 0 & -26 & -13 \\ 0 & -12 & -7 \end{vmatrix}$$

no effect
no change

$$-\frac{1}{26} R_2 \rightarrow R_2 \quad \begin{vmatrix} 1 & 9 & 3 \\ 0 & 1 & 1/2 \\ 0 & -12 & -7 \end{vmatrix}$$

Introduce a factor of $-\frac{1}{26}$

$$12R_2 + R_3 \rightarrow R_3 \quad \begin{vmatrix} 1 & 9 & 3 \\ 0 & 1 & 1/2 \\ 0 & 0 & -1 \end{vmatrix}$$

No change

$$\therefore \begin{vmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{vmatrix} = \underbrace{(-1)}_{\text{undo sign change}} \underbrace{(-26)}_{\text{undo } -\frac{1}{26}} (1 \cdot 1 \cdot -1) = -26$$

$A =$ Determine $\begin{vmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 9 & 2 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{vmatrix}$ after first manipulating the matrix into upper triangular form.

	Row Operation		Effect
$\begin{vmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 9 & 2 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{vmatrix}$	$R_1 \leftrightarrow R_2$	$\begin{vmatrix} 1 & 0 & 9 & 2 \\ 3 & 1 & -1 & 4 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{vmatrix}$	Sign change
	$-3R_1 + R_2 \to R_2$ $-8R_1 + R_3 \to R_3$ $-4R_1 + R_4 \to R_4$	$\begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & -3 & -71 & -11 \\ 0 & 2 & -30 & -9 \end{vmatrix}$	no change
	$3R_2 + R_3 \to R_3$ $-2R_2 + R_4 \to R_4$	$\begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 0 & -155 & -17 \\ 0 & 0 & 26 & -5 \end{vmatrix}$	No change
	$\frac{26}{155}R_3 + R_4 \to R_4$	$\begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 0 & -155 & -17 \\ 0 & 0 & 0 & -233/31 \end{vmatrix}$	no change

$$\therefore \quad \det(A) = -1\left(1 \cdot 1 \cdot -155 \cdot \frac{-233}{31}\right)$$

$$= -1165$$

A couple of things that are true about determinants

If A and B are like-sized square matrices, then $\det(AB) = \det(A)\det(B)$.

If A is any square matrix, then $\det(A^{\mathrm{T}}) = \det(A)$.

Example

Prove the first thing is true for two by two matrices using $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

$$\det(AB) = \begin{vmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{vmatrix}$$

$$= (ax+bz)(cy+dw) - (cx+dz)(ay+bw)$$

$$= \cancel{acxy} + adwx + bcyz + \cancel{bdwz} - \cancel{acxy} - bcwx - adyz - \cancel{bdwz}$$

$$= adwx + bcyz - bcwx - adyz$$

$$= adwx - adyz + bcyz - bcwx$$

$$= ad(wx - yz) - bc(wx - yz) = (ad-bc)(wx-yz)$$

$$= \det(A)\det(B)$$

A Little Geometry

$\left|\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right|$ finds the area of any parallelogram whose sides are parallel to $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

Example

Find the area of the parallelogram outline in Figure 1.

$$\vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

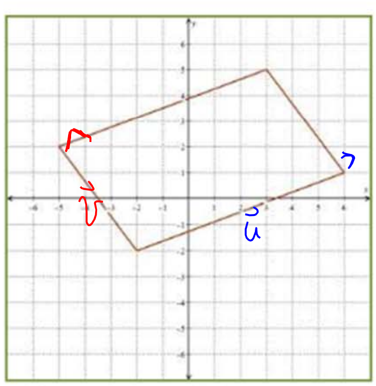

Figure 1: A parallelogram

$$\text{Area} = \left|\begin{vmatrix} -3 & 8 \\ 4 & 3 \end{vmatrix}\right|$$

$$= 41$$

Example

The determinant of one of the objects below is self-evidently zero; "self-evidently" means that the determination can be made via a "trivial" observation without altering the object in any way. Which object satisfies this property?

a. $$\begin{bmatrix} 1 & 1 & 15 & 22 & 29 & 36 & 43 \\ 2 & 1 & 16 & 23 & 30 & 37 & 44 \\ 3 & 1 & 17 & 24 & 31 & 38 & 45 \\ 4 & 1 & 18 & 25 & 32 & 39 & 46 \\ 5 & 1 & 19 & 26 & 33 & 40 & 47 \\ 6 & 1 & 20 & 27 & 34 & 41 & 48 \\ 7 & 1 & 21 & 28 & 35 & 42 & 49 \end{bmatrix}$$

b. 

http://kamiichan.deviantart.com/art/Dalek-328555754

c. $$\begin{bmatrix} 1 & 8 & 15 & 22 & 29 & 36 & 43 \\ 2 & 9 & 16 & 23 & 30 & 37 & 44 \\ 3 & 10 & 17 & 24 & 31 & 38 & 45 \\ 4 & 11 & 18 & 25 & 32 & 39 & 46 \\ 5 & 12 & 19 & 26 & 33 & 40 & 47 \\ 6 & 13 & 20 & 27 & 34 & 41 & 48 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 \end{bmatrix}$$

d. $$\begin{bmatrix} 1 & 2 & 15 & 22 & 29 & 36 & 43 \\ 2 & 4 & 16 & 23 & 30 & 37 & 44 \\ 3 & 6 & 17 & 24 & 31 & 38 & 45 \\ 4 & 8 & 18 & 25 & 32 & 39 & 46 \\ 5 & 10 & 19 & 26 & 33 & 40 & 47 \\ 6 & 12 & 20 & 27 & 34 & 41 & 48 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 \end{bmatrix}$$

A whole lot of equivalent properties

If A is an n x n matrix, then either each of the following statements is true about A or each of the following statements is false about A.

a. A is an invertible matrix (i.e., A is ***nonsingular***).

b. A is row equivalent to I_n.

c. A has n pivot columns.

d. The only solution to $A\vec{x} = \vec{0}$ is $\vec{0}$ (***the trivial solution***).

e. The columns of A form a linearly independent set.

f. The linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one.

g. The equation $A\vec{x} = \vec{b}$ has exactly one solution $\forall\, \vec{b} \in \mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $T(\vec{x}) = A\vec{x}$ is onto $\mathbb{R}^n$.

l. A^{T} is nonsingular.

m. $\det(A) \neq 0$

A fact about inverse matrices

While discussing many algebraic structures, the idea of "left inverses" and "right inverses" comes up. For example, we say that b is a left inverse of a if ba equals the identity element for the structure. In fact, in more advance linear algebra classes we talk about left and right inverse matrices of non-square matrices.

We don't make that distinction for square matrices because either a square matrix has no inverse of any kind (we call such matrices ***singular***) or the left and right inverse matrices are in fact one and the same matrix. That is:

If A is a *square* matrix, then the only way that either $AB = I$ or $BA = I$ is if $B = A^{-1}$.