Determinants (of square matrices)

$$\det([a_{11}]) = |a_{11}| = a_{11}$$

For a square matrix, A, with two or more rows we define the cofactor of entry $\underline{a_{ij}}$, C_{ij}, to be $(-1)^{i+j}$ times the determinant of the matrix that results from eliminating the i^{th} row and j^{th} column from A. Then using any row of A or any column of A:

$$\det(A) = \sum_{j=1}^{n}\left[a_{ij}C_{ij}\right] = \sum_{i=1}^{n}\left[a_{ij}C_{ij}\right]$$

Please note that in the first formula we are summing along a fixed i^{th} row of A whereas in the second formula we are summing along a fixed j^{th} column of A.

Example

Find a simplified formula for $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ - first by summing along the first row and again by summing along the second column.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sum_{j=1}^{2}\left[a_{1,j}C_{1,j}\right]$$

$$= a_{11}C_{11} + a_{12}C_{12}$$

$$= a_{11}(-1)^{1+1}|a_{22}| + a_{12}(-1)^{1+2}|a_{21}|$$

$$= a_{11}(1)a_{22} + a_{12}(-1)a_{21}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sum_{i=1}^{2}\left[a_{i,2}C_{i,2}\right]$$

$$= a_{12}C_{12} + a_{22}C_{22}$$

$$= a_{12}(-1)^{3}|a_{21}| + a_{22}(-1)^{4}|a_{11}|$$

$$= a_{12}(-1)a_{21} + a_{22}(1)a_{11}$$

$$= a_{11}a_{22} - a_{12}a_{21} \checkmark$$

Example

Use cofactors along the second row to find $\det(A)$ where $A = \begin{bmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{bmatrix}$. Verify the determinant value by using cofactors along the first column.

$$\det(A) = \sum_{j=1}^{3} \left[a_{2,j}\, C_{2,j} \right]$$

$$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= a_{21}(-1)^3 \begin{vmatrix} 6 & -1 \\ 9 & 3 \end{vmatrix} + a_{22}(-1)^4 \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + a_{23}(-1)^5 \cdot \begin{vmatrix} 2 & 6 \\ 1 & 9 \end{vmatrix}$$

$$= (3)(-1)(18+9) + (1)(1)(6+1) + (-4)(-1)(18-6)$$

$$= -81 + 7 + 48$$

$$= -26$$

$$\det(A) = \sum_{i=1}^{3} \left[a_{i,1}\, C_{i,1} \right]$$

$$= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= a_{11}(-1)^2 \begin{vmatrix} 1 & -4 \\ 9 & 3 \end{vmatrix} + a_{21}(-1)^3 \begin{vmatrix} 6 & -1 \\ 9 & 3 \end{vmatrix} + a_{31}(-1)^4 \begin{vmatrix} 6 & -1 \\ 1 & -4 \end{vmatrix}$$

$$= 2(1)(3+36) + (3)(-1)(18+9) + (1)(1)(-24+1)$$

$$= 78 - 81 - 23$$

$$= -26$$

Example: Evaluate the determinate; the equal sign has been introduced to save space.

$$\begin{vmatrix} 3 & 9 & 0 & -1 \\ 0 & -3 & -2 & 7 \\ 2 & 5 & 0 & 4 \\ 0 & -6 & 0 & 6 \end{vmatrix} = \sum_{i=1}^{4} \left[a_{i,3} C_{i,3} \right]$$

$$= a_{13} C_{13} + a_{23} C_{23} + a_{33} C_{33} + a_{43} C_{43}$$

$$= 0 + a_{23}(-1)^5 \begin{vmatrix} 3 & 9 & -1 \\ 2 & 5 & 4 \\ 0 & -6 & 6 \end{vmatrix} + 0 + 0$$

$$= a_{23}(-1) \cdot \left[3(-1)^2 \begin{vmatrix} 5 & 4 \\ -6 & 6 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 9 & -1 \\ -6 & 6 \end{vmatrix} + 0 \right]$$

$$= (-2)(-1)\left[(3)(1)(30+24) + 2(-1)(54-6) \right]$$

$$= 132$$

Example

Use a determinant to find $\vec{u} \times \vec{v}$ where $\vec{u} = [1, 7, -3]$ and $\vec{v} = [3, 0, 5]$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 7 & -3 \\ 3 & 0 & 5 \end{vmatrix}$$

$$= \sum_{j=1}^{3} \left[a_{1,j} C_{1,j} \right]$$

$$= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= \hat{i}(-1)^2 \begin{vmatrix} 7 & 3 \\ 0 & 5 \end{vmatrix} + \hat{j}(-1)^3 \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} + \hat{k}(-1)^4 \begin{vmatrix} 1 & 7 \\ 3 & 0 \end{vmatrix}$$

$$= \hat{i}(35 - 0) + \hat{j}(-1)(5+9) + \hat{k}(0 - 21)$$

$$= [35, -14, -21] = 7[5, -2, -3]$$

Elementary Matrices

An ***elementary matrix*** is a matrix that can be created from an identity matrix via one elementary row operation.

Example: Let $B = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$.

For each of the following matrices (each identified as A), describe the row operation that was effected upon I_2 to create A. Then find $|A|$ and compare its value to $|I_2|$. Next, find AB and describe the difference between it and B. Finally, compare the values of $|AB|$ and $|B|$. This example continues on page 5.

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\det(A) = -\det(I_2)$

$\det(A) = 0 - 1$
$= -1$

$\det(I_2) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$
$= 1 - 0$
$= 1$

$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$
$= \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$

$\det(B) = 3 - (-2)$
$= 5$

$B \quad R_1 \leftrightarrow R_2 \quad AB$

$\det(AB) = -2 - 3$
$= -5$
$= -\det(B)$

$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} 3R_1 \rightarrow R_1 \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3 = 3\det(I_2)$

$AB = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$
$= \begin{bmatrix} 9 & -3 \\ 2 & 1 \end{bmatrix}$

$\det(AB) = 9 + 6$
$= 15$
$= 3(5)$
$= 3\det(B)$

$B \quad 3R_1 \rightarrow R_1 \quad AB$

$A=\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$ $\quad \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \xrightarrow{-2R_2 \to R_2} \begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$

$\det(A) = \begin{vmatrix}1 & 0\\ 0 & -2\end{vmatrix} = -2 = -2\det(I_2)$

$AB = \begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix} = \begin{bmatrix}3 & -1\\ -4 & -2\end{bmatrix}$

$B \quad -2R_2 \to R_2 \quad AB$

$\det(AB) = -6 - 4$
$= -10$
$= -2\det(B)$

$A=\begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}$ $\quad \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \xrightarrow{-3R_1+R_2 \to R_2} \begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}$

$\det(A) = \begin{vmatrix}1 & 0\\ -3 & 1\end{vmatrix} = 1 - 0 = 1 = \det(I_2)$

$AB = \begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$
$= \begin{bmatrix}3 & -1\\ -7 & 4\end{bmatrix}$

$\det(AB) = 12 - 7$
$= 5$
$= \det(B)$

$B \quad -3R_1 + R_2 \to R_2 \quad AB$

$A=\begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}$ $\quad \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \xrightarrow{4R_2+R_1 \to R_1} \begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}$

$\det(A) = 1 = \det(I_2)$

$AB = \begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$
$= \begin{bmatrix}11 & 3\\ 2 & 1\end{bmatrix}$

$\det(AB) = 11 - 6$
$= 5$
$= \det(B)$

$B \quad 4R_2 + R_1 \to R_1 \quad AB$

Elementary Row Operations and Determinants

Suppose that A and B are square matrices of equal dimension; suppose further that B can be created from A via a single elementary row operation. Then:

- if the operation is adding a multiple of one row of A to a different row of A, then $\det(B)=\det(A)$
- if the operation is swapping two rows of A, then $\det(B)=-\det(A)$
- if the operation is multiplying a row of A by the real number k, then $\det(B)=k\cdot\det(A)$.

A couple of definitions and a convenient fact

An ***upper triangular matrix*** is a matrix where every entry *below* the main diagonal is zero.

A ***lower triangular matrix*** is a matrix where every entry *above* the main diagonal is zero.

The determinant of any $n\times n$ triangular matrix B is given by the formula $\det(B)=\prod_{i=1}^{n} b_{ii}$.

Example

Determine $\begin{vmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{vmatrix}$ after first manipulating the matrix into upper triangular form.

det

$\begin{vmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{vmatrix}$ $R_1 \leftrightarrow R_3$ $\begin{vmatrix} 1 & 9 & 3 \\ 3 & 1 & -4 \\ 2 & 6 & -1 \end{vmatrix}$ -1

$-3R_1+R_2\to R_2$
$-2R_1+R_3\to R_3$ $\begin{vmatrix} 1 & 9 & 3 \\ 0 & -26 & -13 \\ 0 & -12 & -7 \end{vmatrix}$ no change

$-\frac{1}{26}R_2\to R_2$ $\begin{vmatrix} 1 & 9 & 3 \\ 0 & 1 & 1/2 \\ 0 & -12 & -7 \end{vmatrix}$ $-\frac{1}{26}$

$12R_2+R_3\to R_3$ $\begin{vmatrix} 1 & 9 & 3 \\ 0 & 1 & 1/2 \\ 0 & 0 & -1 \end{vmatrix}$ no change

$\therefore \begin{vmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{vmatrix} = (-1)(-26)\cdot[(1)(1)(-1)]$

$= -26$

Determine $\begin{vmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 9 & 2 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{vmatrix}$ after first manipulating the matrix into upper triangular form.

det

$$\begin{vmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 9 & 2 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{vmatrix} \quad \begin{matrix} R_1 \leftrightarrow R_2 \\ R_3 \leftrightarrow R_4 \end{matrix} \quad \begin{vmatrix} 1 & 0 & 9 & 2 \\ 3 & 1 & -1 & 4 \\ 4 & 2 & 6 & -1 \\ 8 & -3 & 1 & 7 \end{vmatrix}$$

no change

$$\begin{matrix} -3R_1 + R_2 \to R_2 \\ -4R_1 + R_3 \to R_3 \\ -8R_1 + R_4 \to R_4 \end{matrix} \quad \begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 2 & -30 & -9 \\ 0 & -3 & -71 & -9 \end{vmatrix}$$

no change

$$\begin{matrix} -2R_2 + R_3 \to R_3 \\ 3R_2 + R_4 \to R_4 \end{matrix} \quad \begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 0 & 26 & -5 \\ 0 & 0 & -155 & -15 \end{vmatrix}$$

no change

$$\begin{matrix} \frac{1}{26} R_3 \to R_3 \\ -\frac{1}{5} R_4 \to R_4 \end{matrix} \quad \begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 0 & 1 & -5/26 \\ 0 & 0 & 31 & 3 \end{vmatrix}$$

$\left(\frac{1}{26}\right)\left(-\frac{1}{5}\right)$

$$-31R_3 + R_4 \to R_4 \quad \begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 0 & 1 & -5/26 \\ 0 & 0 & 0 & \frac{233}{26} \end{vmatrix}$$

$$\therefore \begin{vmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 9 & 2 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{vmatrix} = (26)(-5)\left[(1)(1)(1)\left(\frac{233}{26}\right)\right]$$

$$= -1165$$

A couple of things that are true about determinants

If A and B are like-sized square matrices, then $\det(AB) = \det(A)\det(B)$.

If A is any square matrix, then $\det\left(A^{\mathrm{T}}\right) = \det(A)$.

Example

Prove the first thing is true for two by two matrices using $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

$$\det(A)\det(B) = (ad - bc)(xw - yz)$$
$$= adxw - adyz - bcxw + bcyz$$
$$= (adxw + bcyz) - (adyz + bcxw)$$

$$\det(AB) = \det\left(\begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix}\right)$$
$$= (\cancel{axcy} + axdw + bzcy + \cancel{bzdw}) - (\cancel{aycx} + aydz + bwcx + \cancel{bwdz})$$
$$= (adxw + bcyz) - (adyz + bcxw)$$

A Little Geometry

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ finds the area of any parallelogram whose sides are parallel to $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

Example

Find the area of the parallelogram outline in Figure 1.

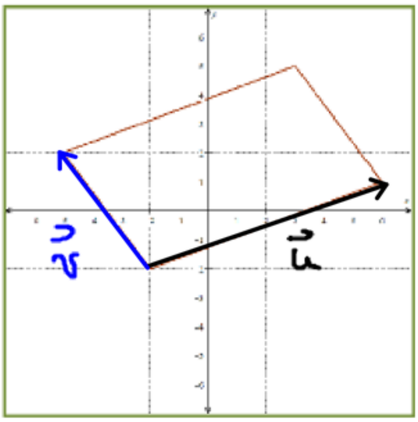


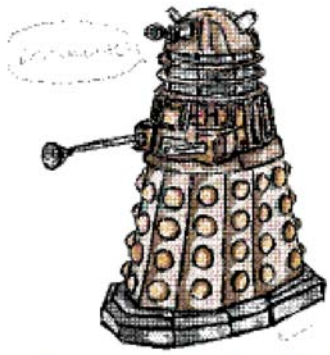
Figure 1: A parallelogram

$$\text{Area} = \left| \vec{u}, \vec{v} \right|$$
$$= \begin{vmatrix} 8 & -3 \\ 3 & 4 \end{vmatrix}$$
$$= 41$$

Example

The determinant of one of the objects below is self-evidently zero; "self-evidently" means that the determination can be made via a "trivial" observation without altering the object in any way. Which object satisfies this property?

a. $\begin{bmatrix} 1 & 1 & 15 & 22 & 29 & 36 & 43 \\ 2 & 1 & 16 & 23 & 30 & 37 & 44 \\ 3 & 1 & 17 & 24 & 31 & 38 & 45 \\ 4 & 1 & 18 & 25 & 32 & 39 & 46 \\ 5 & 1 & 19 & 26 & 33 & 40 & 47 \\ 6 & 1 & 20 & 27 & 34 & 41 & 48 \\ 7 & 1 & 21 & 28 & 35 & 42 & 49 \end{bmatrix}$

b. 

http://kamiichan.deviantart.com/art/Dalek-328555754

c. $\begin{bmatrix} 1 & 8 & 15 & 22 & 29 & 36 & 43 \\ 2 & 9 & 16 & 23 & 30 & 37 & 44 \\ 3 & 10 & 17 & 24 & 31 & 38 & 45 \\ 4 & 11 & 18 & 25 & 32 & 39 & 46 \\ 5 & 12 & 19 & 26 & 33 & 40 & 47 \\ 6 & 13 & 20 & 27 & 34 & 41 & 48 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 2 & 15 & 22 & 29 & 36 & 43 \\ 2 & 4 & 16 & 23 & 30 & 37 & 44 \\ 3 & 6 & 17 & 24 & 31 & 38 & 45 \\ 4 & 8 & 18 & 25 & 32 & 39 & 46 \\ 5 & 10 & 19 & 26 & 33 & 40 & 47 \\ 6 & 12 & 20 & 27 & 34 & 41 & 48 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 \end{bmatrix}$

A whole lot of equivalent properties

If A is an n x n matrix, then either each of the following statements is true about A or each of the following statements is false about A.

a. A is an invertible matrix (i.e., A is ***nonsingular***).

b. A is row equivalent to I_n.

c. A has n pivot columns.

d. The only solution to $A\vec{x}=\vec{0}$ is $\vec{0}$ (***the trivial solution***).

e. The columns of A form a linearly independent set.

f. The linear transformation $T(\vec{x})=A\vec{x}$ is one-to-one.

g. The equation $A\vec{x}=\vec{b}$ has exactly one solution $\forall\, \vec{b}\in\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $T(\vec{x})=A\vec{x}$ is onto $\mathbb{R}^n$.

l. A^{T} is nonsingular.

m. $\det(A)\neq 0$

A fact about inverse matrices

While discussing many algebraic structures, the idea of "left inverses" and "right inverses" comes up. For example, we say that b is a left inverse of a if ba equals the identity element for the structure. In fact, in more advance linear algebra classes we talk about left and right inverse matrices of non-square matrices.

We don't make that distinction for square matrices because either a square matrix has no inverse of any kind (we call such matrices ***singular***) or the left and right inverse matrices are in fact one and the same matrix. That is:

If A is a *square* matrix, then the only way that either $AB=I$ or $BA=I$ is if $B=A^{-1}$.