determinant bars (not absolute value)

Determinants (of square matrices)

$$\det\left(\left[a_{11}\right]\right) = \left|\, a_{11} \,\right| = a_{11}$$

For a square matrix, A, with two or more rows we define the cofactor of entry $\underline{a_{ij}}$, C_{ij}, to be $(-1)^{i+j}$ times the determinant of the matrix that results from eliminating the i^{th} row and j^{th} column from A. Then using any row of A or any column of A:

$$\det(\mathrm{A}) = \sum_{j=1}^{n}\left[a_{ij}\,C_{ij}\right] = \sum_{i=1}^{n}\left[a_{ij}\,C_{ij}\right]$$

Please note that in the first formula we are summing along a fixed i^{th} row of A whereas in the second formula we are summing along a fixed j^{th} column of A.

Example

Find a simplified formula for $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ - first by summing along the first row and again by summing along the second column.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sum_{j=1}^{2}\left[a_{1,j}\,C_{1,j}\right]$$

$$= a_{11}C_{11} + a_{12}C_{12}$$

$$= a_{11}\left[(-1)^{1+1}\left|a_{22}\right|\right] + a_{12}\left[(-1)^{1+2}\left|a_{21}\right|\right]$$

$$= a_{11}(1)a_{22} + a_{12}(-1)a_{21}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sum_{i=1}^{2}\left[a_{i,2}\,C_{i,2}\right]$$

$$= a_{12}C_{12} + a_{22}C_{22}$$

$$= a_{12}\left[(-1)^{1+2}\left|a_{21}\right|\right] + a_{22}\left[(-1)^{2+2}\left|a_{11}\right|\right]$$

$$= a_{12}(-1)a_{21} + a_{22}(1)a_{11}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

Example

Use cofactors along the second row to find $\det(A)$ where $A=\begin{bmatrix}2 & 6 & -1\\ 3 & 1 & -4\\ 1 & 9 & 3\end{bmatrix}$. Verify the determinant value by using cofactors along the first column.

$$\begin{vmatrix}2 & 6 & -1\\ 3 & 1 & -4\\ 1 & 9 & 3\end{vmatrix} = \sum_{j=1}^{3}\left[a_{2,j}\,C_{2,j}\right]$$

$$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= 3\left[(-1)^{2+1}\begin{vmatrix}6 & -1\\ 9 & 3\end{vmatrix}\right] + (1)\left[(-1)^{2+2}\begin{vmatrix}2 & -1\\ 1 & 3\end{vmatrix}\right] + (-4)\left[(-1)^{2+3}\begin{vmatrix}2 & 6\\ 1 & 9\end{vmatrix}\right]$$

$$= 3(-1)(18-(-9)) + (1)(1)(6-(-1)) + (-4)(-1)(18-6)$$

$$= -81 + 7 + 48$$

$$= -26$$

$$\begin{vmatrix}2 & 6 & -1\\ 3 & 1 & -4\\ 1 & 9 & 3\end{vmatrix} = \sum_{i=1}^{3}\left[a_{i,1}\,C_{i,1}\right]$$

$$= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= (2)(-1)^{1+1}\begin{vmatrix}1 & -4\\ 9 & 3\end{vmatrix} + (3)(-1)^{1+2}\begin{vmatrix}6 & -1\\ 9 & 3\end{vmatrix} + (1)(-1)^{1+3}\begin{vmatrix}6 & -1\\ 1 & -4\end{vmatrix}$$

$$= (2)(1)(3-(-36)) + 3(-1)(18-(-9)) + (1)(1)(-24-(-1))$$

$$= 78 - 81 - 23$$

$$= -26$$

Example: Evaluate the determinate; the equal sign has been introduced to save space.

$$\begin{vmatrix} 3 & 9 & 0 & -1 \\ 0 & -3 & -2 & 7 \\ 2 & 5 & 0 & 4 \\ 0 & -6 & 0 & 6 \end{vmatrix} = \sum_{i=1}^{4} \left[a_{i,3}\, C_{i,3} \right]$$

$$= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43}$$

$$= 0\,C_{13} + (-2)(-1)^{2+3} \begin{vmatrix} 3 & 9 & -1 \\ 2 & 5 & 4 \\ 0 & -6 & 6 \end{vmatrix} + 0C_{33} + 0\,C_{43}$$

$$= (+2)\left[(3)(-1)^{1+1} \begin{vmatrix} 5 & 4 \\ -6 & 6 \end{vmatrix} + 2(-1)^{2+1} \begin{vmatrix} 9 & -1 \\ -6 & 6 \end{vmatrix} + 0 \right]$$

$$= 2\left[3(54) + (-2)(48) \right]$$

$$= 132$$

Example

Use a determinant to find $\vec{u} \times \vec{v}$ where $\vec{u} = [1, 7, -3]$ and $\vec{v} = [3, 0, 5]$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 7 & -3 \\ 3 & 0 & 5 \end{vmatrix}$$

$$= \hat{i}(-1)^{1+1} \begin{vmatrix} 7 & -3 \\ 0 & 5 \end{vmatrix} + \hat{j}(-1)^{1+2} \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} + \hat{k}(-1)^{1+3} \begin{vmatrix} 1 & 7 \\ 3 & 0 \end{vmatrix}$$

$$= \hat{i}(35 - 0) - \hat{j}(5 - (-9)) + \hat{k}(0 - 21)$$

$$= 35\hat{i} - 14\hat{j} - 21\hat{k}$$

$\det(B) = 3-(-2)$
$= 5$

Elementary Matrices

An ***elementary matrix*** is a matrix that can be created from an identity matrix via one elementary row operation.

Example: Let $B = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

For each of the following matrices (each identified as A), describe the row operation that was effected upon I_2 to create A. Then find $|A|$ and compare its value to $|I_2|$. Next, find AB and describe the difference between it and B. Finally, compare the values of $|AB|$ and $|B|$. This example continues on page 5.

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$I_2 \xrightarrow{R_1 \leftrightarrow R_2} A$

$\det(A) = 0 - 1$
$= -1$
$= -\det(I_2)$

$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$
$= \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$

$B \xrightarrow{R_1 \leftrightarrow R_2} AB$

$\det(AB) = -2-3$
$= -5$
$= -\det(B)$

$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

① $I_2 \xrightarrow{3R_1 \to R_1} A$

② $\det(A) = 3$
$= 3\det(I_2)$

③ $AB = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$
$= \begin{bmatrix} 9 & -3 \\ 2 & 1 \end{bmatrix}$

④ $B \xrightarrow{3R_1 \to R_1} AB$

⑤ $\det(AB) = 9-(-6)$
$= 15$
$= 3\det(B)$

$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ $\quad I_2 \quad -2R_2 \to R_2 \quad A$

$\det(A) = -2$
$= -2\det(I_2)$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$
$= \begin{bmatrix} 3 & -1 \\ -4 & -2 \end{bmatrix}$

$B \quad -2R_2 \to R_2 \quad AB$

$\det(AB) = -6 - 4$
$= -10$
$= -2\det(B)$

$A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $\quad I_2 \quad -3R_1 + R_2 \to R_2 \quad A$

$\det(A) = 1$
$= \det(I_2)!$

$AB = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$
$= \begin{bmatrix} 3 & -1 \\ -7 & 4 \end{bmatrix}$

$B \quad -3R_1 + R_2 \to R_2 \quad AB$

$\det(AB) = 12 - 7$
$= 5$
$= \det(B)$

$A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ $\quad I_2 \quad 4R_2 + R_1 \to R_1 \quad A$

$\det(A) = 1$
$= \det(I_2)$

$AB = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$
$= \begin{bmatrix} 11 & 3 \\ 2 & 1 \end{bmatrix}$

$B \quad 4R_2 + R_1 \to R_1 \quad AB$

$\det(AB) = 11 - 6$
$= 5$
$= \det(B)!$

Elementary Row Operations and Determinants

Suppose that A and B are square matrices of equal dimension; suppose further that B can be created from A via a single elementary row operation. Then:

- if the operation is adding a multiple of one row of A to a different row of A, then $\det(\mathrm{B}) = \det(\mathrm{A})$
- if the operation is swapping two rows of A, then $\det(\mathrm{B}) = -\det(\mathrm{A})$
- if the operation is multiplying a row of A by the real number k, then $\det(\mathrm{B}) = k \cdot \det(\mathrm{A})$.

A couple of definitions and a convenient fact

An ***upper triangular matrix*** is a matrix where every entry *below* the main diagonal is zero.
A ***lower triangular matrix*** is a matrix where every entry *above* the main diagonal is zero.

The determinant of any $\mathbf{n} \times \mathbf{n}$ triangular matrix B is given by the formula $\det(\mathrm{B}) = \prod^{n} b_{ii}$.

Example

Determine $\begin{vmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{vmatrix}$ after first manipulating the matrix into upper triangular form.

$$\begin{vmatrix} 2 & 6 & -1 \\ 3 & 1 & -4 \\ 1 & 9 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 9 & 3 \\ 3 & 1 & -4 \\ 2 & 6 & -1 \end{vmatrix} \qquad R_1 \leftrightarrow R_3$$

$$= -\begin{vmatrix} 1 & 9 & 3 \\ 0 & -26 & -13 \\ 0 & -12 & -7 \end{vmatrix} \qquad \begin{matrix} -3R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{matrix}$$

$$= -(-13)\begin{vmatrix} 1 & 9 & 3 \\ 0 & 2 & 1 \\ 0 & -12 & -7 \end{vmatrix} \qquad -\frac{1}{13}R_2 \to R_2$$

$$= 13\begin{vmatrix} 1 & 9 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{vmatrix} \qquad 6R_2 + R_3 \to R_3$$

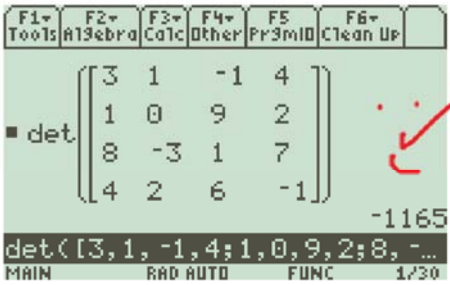
$$= 13(1)(2)(-1)$$

$$= -26$$

Determine $\begin{vmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 9 & 2 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{vmatrix}$ after first manipulating the matrix into upper triangular form.

$$\begin{vmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 9 & 2 \\ 8 & -3 & 1 & 7 \\ 4 & 2 & 6 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 9 & 2 \\ 3 & 1 & -1 & 4 \\ 4 & 2 & 6 & -1 \\ 8 & -3 & 1 & 7 \end{vmatrix} \qquad \begin{array}{l} R_1 \leftrightarrow R_2 \ (-) \\ R_3 \leftrightarrow R_4 \ (-) \end{array}$$

$$= \begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 2 & -30 & -9 \\ 0 & -3 & -71 & -9 \end{vmatrix} \qquad \begin{array}{l} -3R_1 + R_2 \to R_2 \\ -4R_1 + R_3 \to R_3 \\ -8R_1 + R_4 \to R_4 \end{array}$$



$$= \begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 0 & 26 & -5 \\ 0 & 0 & -155 & -15 \end{vmatrix} \qquad \begin{array}{l} -2R_2 + R_3 \to R_3 \\ 3R_2 + R_4 \to R_4 \end{array}$$

$$\det(A) = (1)(1)\begin{vmatrix} 26 & -5 \\ -155 & -15 \end{vmatrix}$$

$$= 26(-15) - (-5)(-155)\ldots$$

$$= -1165$$

$$= 26\begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 0 & 1 & -5/26 \\ 0 & 0 & -155 & -15 \end{vmatrix} \qquad \tfrac{1}{26}R_3 \to R_3$$

$$= 26\begin{vmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -28 & -2 \\ 0 & 0 & 1 & -5/26 \\ 0 & 0 & 0 & -1165/26 \end{vmatrix} \qquad 155R_3 + R_4 \to R_4$$

$$= 26(1)(1)(1)(-1165/26)$$

$$= -1165$$

A couple of things that are true about determinants

If A and B are like-sized square matrices, then $\det(\mathrm{AB}) = \det(\mathrm{A})\det(\mathrm{B})$.

If A is any square matrix, then $\det\left(\mathrm{A}^{\mathrm{T}}\right) = \det(\mathrm{A})$.

Example

Prove the first thing is true for two by two matrices using $\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

$$\det(AB) = \det\left(\begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix}\right)$$

$$= (ax+bz)(cy+dw) - (cx+dz)(ay+bw)$$

$$= \cancel{acxy} + adwx + bcyz + \cancel{bdwz}$$
$$- \cancel{acxy} - bcwx - adyz - \cancel{bdwz}$$

$$= adwx - adyz + bcyz - bcwx$$
$$= ad(wx - yz) - bc(wx - yz)$$
$$= (ad - bc)(wx - yz) = \det(A)\det(B)$$

A Little Geometry

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ finds the area of any parallelogram whose sides are parallel to $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

Example

Find the area of the parallelogram outline in Figure 1.

$|\det(A)|$

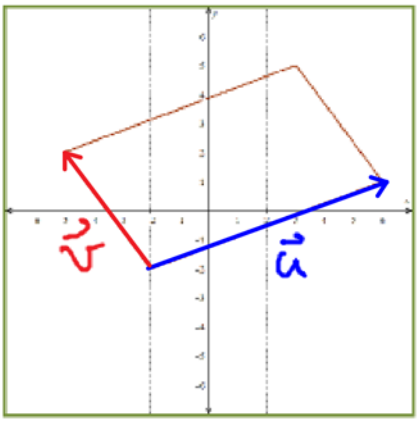


Figure 1: A parallelogram

$\vec{u} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$$\text{Area} = \left|\begin{vmatrix} 8 & -3 \\ 3 & 4 \end{vmatrix}\right| \qquad \begin{vmatrix} -3 & 8 \\ 4 & 3 \end{vmatrix}$$

$$= |32 + 9| \qquad -9 - 32$$

$$= 41 \qquad = -41$$