Transformations

A ***transformation***, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function that assigns to each vector in $\mathbb{R}^n$ a unique vector in $\mathbb{R}^m$. If $T\left(\vec{x}\right)=\vec{b}$, we say that $\vec{b}$ is the ***image*** of $\vec{x}$ under T.

$\mathbb{R}^n$ is called the ***domain*** of T and $\mathbb{R}^m$ is called the ***codomain*** of T. The set of all images found under T is called the ***range*** of T.

Example

Suppose that T is the transformation defined by the rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)=\begin{bmatrix} x_2 \\ 0 \end{bmatrix}$. What are the domain, codomain, and range of T? What is the image of $\vec{x}$ where $\vec{x}=\begin{bmatrix} 5 & -2 & -7 \end{bmatrix}^{\mathrm{T}}$? Describe the set of vectors whose images are $\vec{0}$.

Linear Transformations

A ***linear transformation***, $T:\mathbb{R}^n \to \mathbb{R}^m$, is a transformation that satisfies both of the following properties.

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \text{ and } T(c\vec{u}) = cT(\vec{u}) \quad \forall \vec{u},\vec{v} \in \mathbb{R}^n \text{ and } c \in \mathbb{R}$$

Example

Show that $T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$ is a linear transformation whereas $T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 1 \end{bmatrix}$ is not.

Example

Draw $T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$ given that T is a linear transformation and the images for $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are those shown in Figure 1.

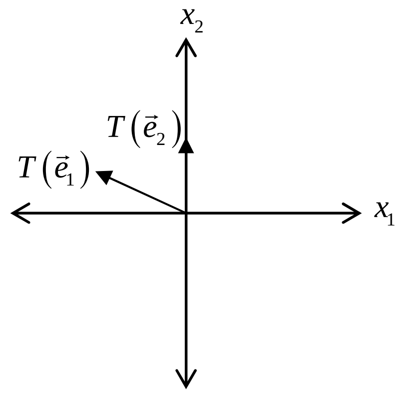

Figure 1: Transformation Vectors

A definition and a very convenient fact

The identity matrix, I_n, is the n x n matrix that has 1s for every entry along the main diagonal and 0 for every other entry.

If we let $\vec{e}_i$ represent the i^{th} column of I_n, then the images of $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ under the linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ completely determines **<u>all</u>** of the images under T.

Example

Suppose that $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and that $T(\vec{e}_1) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

a. Determine $T\left(\begin{bmatrix} -6 & 2 & 1 \end{bmatrix}^{\mathrm{T}}\right)$.

$T(\vec{e}_1) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

b. Find a matrix, M, with the property that $T(\vec{x}) = M\,\vec{x} \;\; \forall \;\; \vec{x} \in \mathbb{R}^3$.

Theorem

Every transformation of form $T(\vec{x}) = A\vec{x}$ is a linear transformation and if T is a linear transformation there exists a unique matrix A such that $T(\vec{x}) = A\vec{x}$.

Example

Suppose that $T:\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation. Find the matrix for T if $T(\vec{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

Example

Show that $T\left(\vec{x}\right)=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\vec{x}$ is a linear transformation.

Example

Find a matrix A with the property that $T\left(\vec{x}\right) = A\vec{x}$ rotates each vector in the $x_1 x_2$-plane by 60° in the counter-clockwise direction. Illustrate the effect of the transformation on the "unit square" shown in Figure 2.

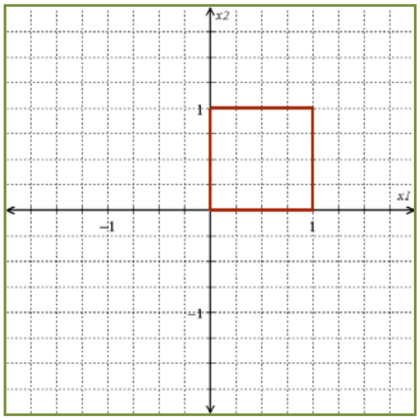

Figure 2: Rotated "unit square"

Definitions and a Theorem

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** $\mathbb{R}^m$ if and only if the range of the transformation is $\mathbb{R}^m$; that is, the transformation is onto $\mathbb{R}^m$ if and only if every vector in $\mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$.

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if and only if $T(\vec{u}) = T(\vec{v}) \Leftrightarrow \vec{u} = \vec{v}$. It is trivially shown that the transformation is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{0}$.

Example

Let $A = \begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix}$. Determine whether or not the linear transformation $T(\vec{x}) = A\vec{x}$ is onto and also whether or not it is one-to-one.