Transformations

A ***transformation***, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function that assigns to each vector in $\mathbb{R}^n$ a unique vector in $\mathbb{R}^m$. If $T(\vec{x}) = \vec{b}$, we say that $\vec{b}$ is the ***image*** of $\vec{x}$ under T.

$\mathbb{R}^n$ is called the ***domain*** of T and $\mathbb{R}^m$ is called the ***codomain*** of T. The set of all images found under T is called the ***range*** of T.

Example

Suppose that T is the transformation defined by the rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$. What are the domain, codomain, and range of T? What is the image of $\vec{x}$ where $\vec{x} = \begin{bmatrix} 5 & -2 & -7 \end{bmatrix}^T$? Describe the set of vectors whose images are $\vec{0}$.

The domain is $\mathbb{R}^3$, the codomain is $\mathbb{R}^2$. The range is a subset of $\mathbb{R}^2$, specifically $\underbrace{\left\{ \begin{bmatrix} x_2 \\ 0 \end{bmatrix} \middle| x_2 \in \mathbb{R} \right\}}_{\text{Basically "the x-axis"!}}$

$T\left(\begin{bmatrix} 5 \\ -2 \\ -7 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, so the image of $\vec{x}$ under T is $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$\underbrace{\text{If } T(\vec{v}) = \vec{0}, \ \vec{v} \text{ must have form } \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix} \middle| x_1, x_3 \in \mathbb{R}}_{\text{This set is called the Kernel of } T.}$

Loosy-Goosy talk to get a concept across

note

Range	Kernel
x-axis	yz-plane

never mind

T maps $\mathbb{R}^n$ to $\mathbb{R}^m$

Linear Transformations

A ***linear transformation***, $T:\mathbb{R}^n \to \mathbb{R}^m$, is a transformation that satisfies both of the following properties.

$$\underbrace{T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})}_{\text{The image of } \vec{u}+\vec{v} \text{ is the sum of the images of } \vec{u} + \vec{v}.} \text{ and } T(c\,\vec{u}) = c\,T(\vec{u}) \quad \forall \vec{u},\vec{v} \in \mathbb{R}^n \text{ and } c \in \mathbb{R}$$

Example

Show that $T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$ is a linear transformation whereas $T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 1 \end{bmatrix}$ is not.

$$\begin{aligned} T_1\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) &= T_1\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) \\ &= \begin{bmatrix} u_2+v_2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} v_2 \\ 0 \end{bmatrix} \\ &= T_1\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) + T_1\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) \end{aligned}$$

i.e. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

$$\begin{aligned} c\,T_1\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) &= c\begin{bmatrix} u_2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} cu_2 \\ 0 \end{bmatrix} \\ &= T_1\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}\right) \\ &= T_1\left(c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) \end{aligned}$$

i.e. $T_1(c\vec{u}) = cT_1(\vec{u})$

QED

To prove that something is not true for all vectors, you need to only show that it is not true for one vector (or in the case of the sum, one sum)

$$T_2\left(\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} + \begin{bmatrix} 6 \\ 11 \\ 54 \end{bmatrix}\right) = T_2\left(\begin{bmatrix} 7 \\ 15 \\ 63 \end{bmatrix}\right) = \begin{bmatrix} 15 \\ 1 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}\right) + T_2\left(\begin{bmatrix} 6 \\ 11 \\ 54 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 11 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 15 \\ 1 \end{bmatrix}$$

QED

Example

elementary unit vectors

Draw $T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right)$ given that T is a linear transformation and the images for $\vec{e}_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{e}_2=\begin{bmatrix}0\\1\end{bmatrix}$ are those shown in Figure 1.

$$T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right) = T\left((-2)\begin{bmatrix}1\\0\end{bmatrix}+(3)\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$= T(-2\vec{e}_1+3\vec{e}_2)$$
$$= T(-2\vec{e}_1)+T(3\vec{e}_2)$$
$$= -2T(\vec{e}_1)+3T(\vec{e}_2)$$

Because T is a linear transformation

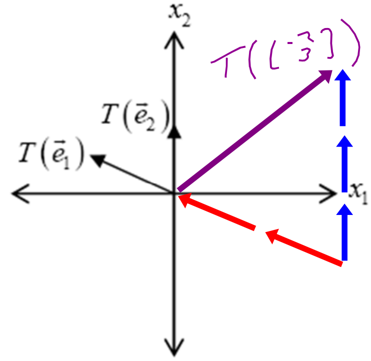


Figure 1: Transformation Vectors

A definition and a very convenient fact

The identity matrix, I_n, is the n x n matrix that has 1s for every entry along the main diagonal and 0 for every other entry.

If we let $\vec{e}_i$ represent the i^{th} column of I_n, then the images of $\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_n$ under the linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$ completely determines **<u>all</u>** of the images under T.

Example

Suppose that $T:\mathbb{R}^3\to\mathbb{R}^2$ and that $T(\vec{e}_1)=\begin{bmatrix}3\\-2\end{bmatrix}$, $T(\vec{e}_2)=\begin{bmatrix}-1\\4\end{bmatrix}$, and $T(\vec{e}_3)=\begin{bmatrix}5\\0\end{bmatrix}$.

$$-6\vec{e}_1 = -6\begin{bmatrix}1\\0\\0\end{bmatrix}$$
$$2\vec{e}_2 = 2\begin{bmatrix}0\\1\\0\end{bmatrix}$$
$$1\vec{e}_3 = 1\begin{bmatrix}0\\0\\1\end{bmatrix}$$

a. Determine $T\left(\begin{bmatrix}-6 & 2 & 1\end{bmatrix}^{\mathrm{T}}\right)$.

$$T\left(\begin{bmatrix}-6\\2\\1\end{bmatrix}\right) = T(-6\vec{e}_1+2\vec{e}_2+1\vec{e}_3)$$
$$= -6T(\vec{e}_1)+2T(\vec{e}_2)+1T(\vec{e}_3)$$
$$= -6\begin{bmatrix}3\\-2\end{bmatrix}+2\begin{bmatrix}-1\\4\end{bmatrix}+1\begin{bmatrix}5\\0\end{bmatrix}$$
$$= \begin{bmatrix}-15\\20\end{bmatrix}$$

$T(\vec{e}_1) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

for all (for every)

b. Find a matrix, M, with the property that $T(\vec{x}) = M\vec{x} \;\; \forall \;\; \vec{x} \in \mathbb{R}^3$.

$$T\left(\begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix}\right) = -6\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 4 \end{bmatrix} + 1\begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$= \begin{array}{c} \begin{bmatrix} 3 & -1 & 5 \\ -2 & 4 & 0 \end{bmatrix} \\ \begin{array}{ccc} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{array} \end{array} \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix}$$

OMG - linear transformations are just matrix/vector products.

Theorem

Every transformation of form $T(\vec{x}) = A\vec{x}$ is a linear transformation and if T is a linear transformation there exists a unique matrix A such that $T(\vec{x}) = A\vec{x}$.

Example

Suppose that $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Find the matrix for T if $T(\vec{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(x_1\vec{e}_1 + x_2\vec{e}_2\right)$$

$$= T(x_1\vec{e}_1) + T(x_2\vec{e}_2)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2\begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Neato!

Awe-some-O

we can cut to the chase → prove Both properties if we just show that

$T(c\vec{u}+\vec{v}) = c\,T(\vec{u}) + T(\vec{v})$

Example

Show that $T(\vec{x}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\vec{x}$ is a linear transformation.

$$T\left(c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} cu_1+v_1 \\ cu_2+v_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} cu_1+v_1 \\ cu_2+v_2 \end{bmatrix}$$

$$= (cu_1+v_1)\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + (cu_2+v_2)\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

$$= cu_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + v_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + cu_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + v_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

$$= c \cdot u_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + c \cdot u_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + v_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + v_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

$$= c \cdot \left(u_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + u_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}\right) + \left(v_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + v_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}\right)$$

$$= c\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{22} & a_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= c\,T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$$

#

$$T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \approx \begin{bmatrix}0.5\\0.9\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \approx \begin{bmatrix}-0.9\\0.5\end{bmatrix} \quad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) \approx \begin{bmatrix}-0.4\\1.4\end{bmatrix}$$

Example

Find a matrix A with the property that $T(\vec{x}) = A\vec{x}$ rotates each vector in the x_1x_2-plane by 60^0 in the counter-clockwise direction. Illustrate the effect of the transformation on the "unit square" shown in Figure 2.

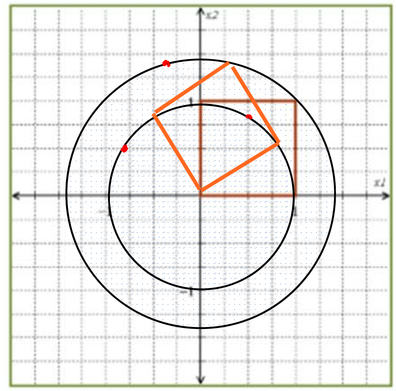
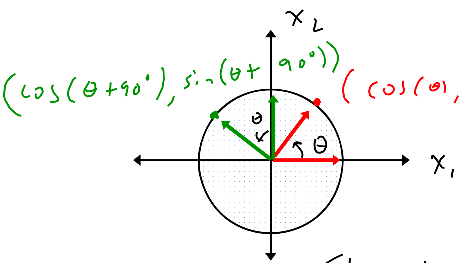


Figure 2: Rotated "unit square"

Deep Background.

rotate by "θ"



The unit circle

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}\cos(\theta+90°)\\ \sin(\theta+90°)\end{bmatrix} = \begin{bmatrix}\cos(\theta)\cos(90°) - \sin(\theta)\sin(90°)\\ \sin(\theta)\cos(90°) + \cos(\theta)\sin(90°)\end{bmatrix} = \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}$$

$\therefore$ $T(\vec{x}) = M\vec{x}$ where $M = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$

Section 1.9 has a boatload of these templates.

On point

$T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix}\cos(60°) & -\sin(60°)\\ \sin(60°) & \cos(60°)\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}$

Trace the corners to see the effect on the square.

$$T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1/2\\ \sqrt{3}/2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sqrt{3}/2\\ 1/2\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}(1-\sqrt{3})/2\\ (\sqrt{3}+1)/2\end{bmatrix}$$

In section 1.9, eg., $T(\vec{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\vec{x}$ affects a reflection across the y-axis. $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{x}$ affects a reflection across the line $y = x$. $T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\vec{x}$ affects a vertical stretch by a factor of 4.

Find a single matrix that takes $\vec{x} \in \mathbb{R}^2$, rotates clockwise by 90°, and then flips the result across the x-axis.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{90^\circ cw} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \xrightarrow{\text{across } x} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{90^\circ cw} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{\text{across } x} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} \cos(-90^\circ) & -\sin(-90^\circ) \\ \sin(-90^\circ) & \cos(-90^\circ) \end{bmatrix}\vec{x}\right)$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\vec{x}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{x} \qquad \text{woo hoo!}$$

If T is both surjective (onto) and injective (one-to-one) it is bijective.

surjective $\left[\quad \right.$

injective $\left[\quad \right.$

$\underline{\text{Deep Background (not about this specific } T \text{)}}$

Observations: $T: \mathbb{R}^3 \to \mathbb{R}^2$ cannot possibly be $\underline{\text{injective}}$

$T: \mathbb{R}^2 \to \mathbb{R}^3$ cannot possibly be $\underline{\text{surjective}}$

$T: \mathbb{R}^3 \to \mathbb{R}^3$ $\underline{\text{can be bijective,}}$ ~~only injective, only surjective~~ or neither injective or surjective.

Face it, $\mathbb{R}^3$ has more vectors than $\mathbb{R}^2$

$\mathbb{R}^3$ has an "∞ number" of vectors and $\mathbb{R}^2$ has "an infinite number" of vectors,

$$\infty_{\mathbb{R}^3} > \infty_{\mathbb{R}^2}$$

Actual Problem

$$T(\vec{x}) = \begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ so } T: \mathbb{R}^3 \to \mathbb{R}^2$$

T cannot possibly be one-to-one (injective)

Is T onto? is there always a solution to $T(\vec{x}) = \vec{b}$ $\forall \vec{b} \in \mathbb{R}^2$?

Can we always solve $x_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} -16 \\ 16 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$?

$$\left[\begin{array}{ccc|c} 3 & -2 & -16 & b_1 \\ 2 & 4 & 16 & b_2 \end{array}\right] \xrightarrow{-\frac{2}{3}R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 3 & -2 & -16 & b_1 \\ 0 & \underbrace{16/3 \quad 80/3}_{\text{cannot be a contradiction}} & & -\frac{2}{3}b_1 + b_2 \end{array}\right] \text{ (REF dude)}$$

Since there's no contradiction, the equation always has a solution, so T is onto $\mathbb{R}^2$.