Transformations

A ***transformation***, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function that assigns to each vector in $\mathbb{R}^n$ a unique vector in $\mathbb{R}^m$. If $T(\vec{x}) = \vec{b}$, we say that $\vec{b}$ is the ***image*** of $\vec{x}$ under T. $\vec{x}$ is a preimage of $\vec{b}$.

$\mathbb{R}^n$ is called the ***domain*** of T and $\mathbb{R}^m$ is called the ***codomain*** of T. The set of all images found under T is called the ***range*** of T.

Example

Suppose that T is the transformation defined by the rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$. What are the domain, codomain, and range of T? What is the image of $\vec{x}$ where $\vec{x} = \begin{bmatrix} 5 & -2 & -7 \end{bmatrix}^T$? Describe the set of vectors whose images are $\vec{0}$.

- The domain of T is $\mathbb{R}^3$, the codomain is $\mathbb{R}^2$, the range is $\left\{ x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \middle| x_2 \in \mathbb{R} \right\}$ (so in fact the range is $\mathbb{R}^1$).

- $T\left(\begin{bmatrix} 5 \\ -2 \\ -7 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

- $T(\vec{x}) = \vec{0} \implies T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\implies \begin{bmatrix} x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies x_2 = 0$$

So the set of preimages of $\vec{0}$ under T have form $\begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

[I have just explicitly shown that every vector that is a preimage of $\vec{0}$ can be written as a linear combination of $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.] $\therefore$ The set of preimages of T is $\operatorname{span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$

★ If T_2 were a linear transformation, then $T\left(7\begin{bmatrix}5\\-3\\6\end{bmatrix}\right)$ and $7T\left(\begin{bmatrix}5\\-3\\6\end{bmatrix}\right)$ would be equal. They're not, so T_2 is not a linear transformation.

 Top 10 concept

Linear Transformations

A ***linear transformation***, $T:\mathbb{R}^n \to \mathbb{R}^m$, is a transformation that satisfies both of the following properties.

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \text{ and } T(c\vec{u}) = cT(\vec{u}) \quad \forall \vec{u},\vec{v} \in \mathbb{R}^n \text{ and } c \in \mathbb{R}$$

Example

Show that $T_1\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_2\\0\end{bmatrix}$ is a linear transformation whereas $T_2\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_2\\1\end{bmatrix}$ is not.

$$T_1(\vec{u}+\vec{v}) = T_1\left(\begin{bmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\end{bmatrix}\right) = \begin{bmatrix}u_2+v_2\\0\end{bmatrix} = \begin{bmatrix}u_2\\0\end{bmatrix} + \begin{bmatrix}v_2\\0\end{bmatrix} = T_1(\vec{u}) + T_1(\vec{v})$$

$$T_1(k\vec{v}) = T_1\left(\begin{bmatrix}kv_1\\kv_2\\kv_3\end{bmatrix}\right) = \begin{bmatrix}kv_2\\0\end{bmatrix} = k\begin{bmatrix}v_2\\0\end{bmatrix} = kT_1(\vec{v})$$

QED

To prove that something is <u>always</u> true you must use variables (as above).

To prove that something is <u>not</u> always true the standard is to show <u>one</u> counterexample.

$$T_2\left(7\begin{bmatrix}5\\-3\\6\end{bmatrix}\right) = T_2\left(\begin{bmatrix}35\\-21\\42\end{bmatrix}\right) = \begin{bmatrix}-21\\1\end{bmatrix}$$

$$7T_2\left(\begin{bmatrix}5\\-3\\6\end{bmatrix}\right) = 7\begin{bmatrix}-3\\1\end{bmatrix} = \begin{bmatrix}-21\\7\end{bmatrix}$$

QED ★

Example

$\hat{i}$ in most books $\hat{j}$ in most books

Draw $T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right)$ given that T is a linear transformation and the images for $\vec{e}_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{e}_2=\begin{bmatrix}0\\1\end{bmatrix}$ are those shown in Figure 1.

$$T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right) = T\left(-2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$= T(-2\vec{e}_1 + 3\vec{e}_2)$$
$$= T(-2\vec{e}_1) + T(3\vec{e}_2)$$
$$= -2T(\vec{e}_1) + 3T(\vec{e}_2)$$

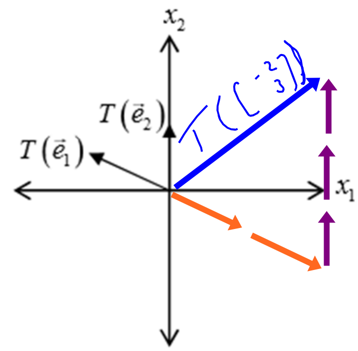


Figure 1: Transformation Vectors

A definition and a very convenient fact

The identity matrix, I_n, is the n x n matrix that has 1s for every entry along the main diagonal and 0 for every other entry.

If we let $\vec{e}_i$ represent the i^{th} column of I_n, then the images of $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ under the linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ completely determines **all** of the images under T.

Example

Suppose that $T:\mathbb{R}^3 \to \mathbb{R}^2$ and that $T(\vec{e}_1)=\begin{bmatrix}3\\-2\end{bmatrix}$, $T(\vec{e}_2)=\begin{bmatrix}-1\\4\end{bmatrix}$, and $T(\vec{e}_3)=\begin{bmatrix}5\\0\end{bmatrix}$. (Assuming T is a linear transformation)

a. Determine $T\left(\begin{bmatrix}-6 & 2 & 1\end{bmatrix}^T\right)$.

$$T\left(\begin{bmatrix}-6\\2\\1\end{bmatrix}\right) = T\left(-6\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$
$$= T(-6\vec{e}_1 + 2\vec{e}_2 + 1\vec{e}_3)$$
$$= -6T(\vec{e}_1) + 2T(\vec{e}_2) + 1T(\vec{e}_3)$$
$$= -6\begin{bmatrix}3\\-2\end{bmatrix} + 2\begin{bmatrix}-1\\4\end{bmatrix} + 1\begin{bmatrix}5\\0\end{bmatrix}$$
$$= \begin{bmatrix}-15\\20\end{bmatrix}$$

$T(\vec{e}_1)=\begin{bmatrix}3\\-2\end{bmatrix}$, $T(\vec{e}_2)=\begin{bmatrix}-1\\4\end{bmatrix}$, and $T(\vec{e}_3)=\begin{bmatrix}5\\0\end{bmatrix}$

b. Find a matrix, M, with the property that $T(\vec{x})=M\vec{x}\ \ \forall\ \ \vec{x}\in\mathbb{R}^3$.

$$T\left(\begin{bmatrix}-6\\2\\1\end{bmatrix}\right) = -6\,T(\vec{e}_1)+2\,T(\vec{e}_2)+1\,T(\vec{e}_3)$$
$$= -6\begin{bmatrix}3\\-2\end{bmatrix}+2\begin{bmatrix}-1\\4\end{bmatrix}+1\begin{bmatrix}5\\0\end{bmatrix}$$
$$= \begin{bmatrix}3 & -1 & 5\\-2 & 4 & 0\end{bmatrix}\begin{bmatrix}-6\\2\\1\end{bmatrix}$$
$$\uparrow T(\vec{e}_1)\quad \uparrow T(\vec{e}_2)\quad \uparrow T(e_3)$$

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \left[\,T(\vec{e}_1)\ \vdots\ T(\vec{e}_2)\ \vdots\ T(\vec{e}_3)\,\right]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Theorem

Every transformation of form $T(\vec{x})=A\vec{x}$ is a linear transformation and if T is a linear transformation there exists a unique matrix A such that $T(\vec{x})=A\vec{x}$.

Example

Suppose that $T:\mathbb{R}^2\to\mathbb{R}^2$ is a linear transformation. Find the matrix for T if $T(\vec{e}_1)=\begin{bmatrix}a_{11}\\a_{21}\end{bmatrix}$ and $T(\vec{e}_2)=\begin{bmatrix}a_{12}\\a_{22}\end{bmatrix}$.

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = T(x_1\vec{e}_1+x_2\vec{e}_2)$$
$$= x_1T(\vec{e}_1)+x_2T(\vec{e}_2)$$
$$= x_1\begin{bmatrix}a_{11}\\a_{21}\end{bmatrix}+x_2\begin{bmatrix}a_{12}\\a_{22}\end{bmatrix}$$
$$= \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$
$$\uparrow T(\vec{e}_1)\quad \uparrow T(\vec{e}_2)$$

Goal: $T(\vec{u})+T(\vec{v})$

Example

Show that $T(\vec{x})=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\vec{x}$ is a linear transformation.

$$T(\vec{u}+\vec{v}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$$

$$= (u_1+v_1)\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + (u_2+v_2)\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

$$= u_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + v_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + u_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + v_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

$$= u_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + u_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + v_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + v_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= T(\vec{u}) + T(\vec{v})$$

Goal: $kT(\vec{u})$

$$T(k\vec{u}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} ku_1 \\ ku_2 \end{bmatrix}$$

$$= ku_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + ku_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

$$= k\left(u_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + u_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}\right)$$

$$= k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= kT(\vec{u})$$

QED

Example

Find a matrix A with the property that $T(\vec{x}) = A\vec{x}$ rotates each vector in the $x_1 x_2$-plane by 60° in the counter-clockwise direction. Illustrate the effect of the transformation on the "unit square" shown in Figure 2.

We can determine A by first determining $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

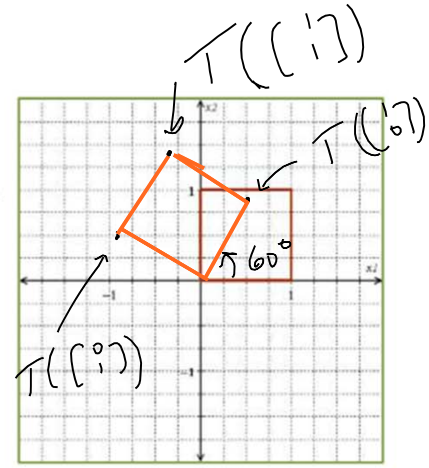
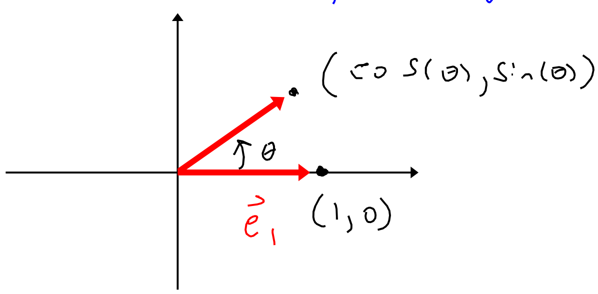
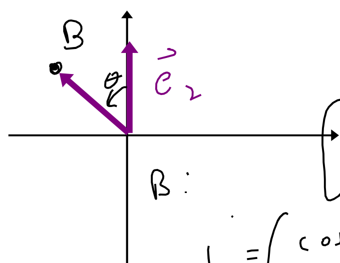


Figure 2: Rotated "unit square"

Let's apply a generic rotation θ.





$$B: \begin{bmatrix}\cos(90^{\circ}+\theta)\\ \sin(90^{\circ}+\theta)\end{bmatrix} = \begin{pmatrix}\cos(90^{\circ})\cos(\theta) - \sin(90^{\circ})\sin(\theta)\\ \sin(90^{\circ})\cos(\theta) + \cos(90^{\circ})\sin(\theta)\end{pmatrix} = \begin{pmatrix}0\cdot\cos(\theta) - 1\cdot\sin(\theta)\\ 1\cdot\cos(\theta) + 0\cdot\sin(\theta)\end{pmatrix} = \begin{pmatrix}-\sin(\theta)\\ \cos(\theta)\end{pmatrix}$$

$$\therefore\ T(\vec{e}_1) = \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix} \text{ and } T(\vec{e}_2) = \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix},$$

So $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$

We did this - you do not ever need to recreate the wheel

For $\theta = 60^{\circ}$

The transformation matrix is

$$A = \begin{bmatrix}\cos(60^{\circ}) & -\sin(60^{\circ})\\ \sin(60^{\circ}) & \cos(60^{\circ})\end{bmatrix} = \begin{bmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}$$ (For computational purposes)

So $T(\vec{e}_1) = \begin{bmatrix}1/2\\ \frac{\sqrt{3}}{2}\end{bmatrix}$

and $T(\vec{e}_2) = \begin{bmatrix}-\sqrt{3}/2\\ 1/2\end{bmatrix}$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1-\sqrt{3}\\ \sqrt{3}+1\end{bmatrix}$$

For graphing purposes: $T(\vec{e}_1) \approx \begin{bmatrix}.5\\.9\end{bmatrix}$, $T(\vec{e}_2) \approx \begin{bmatrix}-.9\\.5\end{bmatrix}$, $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) \approx \begin{bmatrix}-.4\\1.4\end{bmatrix}$

Transformations that are both surjective and injective are called bijective

↳ one-to-one and onto

Definitions and a Theorem

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** $\mathbb{R}^m$ if and only if the range of the transformation is $\mathbb{R}^m$; that is, the transformation is onto $\mathbb{R}^m$ if and only if every vector in $\mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$.

Surjective

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if and only if $T(\vec{u}) = T(\vec{v}) \Leftrightarrow \vec{u} = \vec{v}$. It is trivially shown that the transformation is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{0}$.

Injective

Example

Let $A = \begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix}$. Determine whether or not the linear transformation $T(\vec{x}) = A\vec{x}$ is onto and also whether or not it is one-to-one.

$A\vec{x}$ Rows

2×3 3×1 2×1

$T: \mathbb{R}^3 \to \mathbb{R}^2$

A transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ cannot possibly be one-to-one. If we smush every vector from $\mathbb{R}^3$ onto the xy-plane, inevitably some vectors (points) will lie smack dab on top of other vectors (points). [Similarly, $T_1: \mathbb{R}^2 \to \mathbb{R}^3$ cannot possibly be onto $\mathbb{R}^3$.]

Is T onto $\mathbb{R}^2$?

Is there always at least one solution to $A\vec{x} = \vec{b}$ for any $\vec{b} \in \mathbb{R}^2$

$$\begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Rightarrow x_1\begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 4 \end{bmatrix} + x_3\begin{bmatrix} -16 \\ 16 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -2 & -16 & b_1 \\ 2 & 4 & 16 & b_2 \end{array}\right] \xrightarrow{-\frac{2}{3}R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 3 & -2 & -16 & b_1 \\ 0 & \frac{16}{3} & \frac{80}{3} & b_2 - \frac{2}{3}b_1 \end{array}\right] \text{ (REF)}$$

This resultant system is always consistent, so T is onto $\mathbb{R}^2$

$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$ is onto $\mathbb{R}^2$

T is most definitely not one-to-one.

$T\left(\begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad T\left(\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

two different inputs have the same output

$Q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 12 \\ y \end{bmatrix}$ is not onto $\mathbb{R}^3$

e.g. $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ has no preimage under Q;

nothing maps to $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \leftarrow$ 3 is not 12

Q is one-to-one; changing either

x or y changes $\begin{bmatrix} x \\ 12 \\ y \end{bmatrix}$