Transformations $\vec{x}$ is a preimage of $\vec{b}$

A ***transformation***, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function that assigns to each vector in $\mathbb{R}^n$ a unique vector in $\mathbb{R}^m$. If $T(\vec{x}) = \vec{b}$, we say that $\vec{b}$ is the ***image*** of $\vec{x}$ under T.

$\mathbb{R}^n$ is called the ***domain*** of T and $\mathbb{R}^m$ is called the ***codomain*** of T. The set of all images found under T is called the ***range*** of T.

Example

Suppose that T is the transformation defined by the rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$. What are the domain, codomain, and range of T? What is the image of $\vec{x}$ where $\vec{x} = \begin{bmatrix} 5 & -2 & -7 \end{bmatrix}^T$? Describe the set of vectors whose images are $\vec{0}$.

The Domain of T is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^2$.

The Range of T is not all of $\mathbb{R}^2$

The Range is $\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$ which is, in fact, $\mathbb{R}^1$.

$$T\left(\begin{bmatrix} 5 \\ -2 \\ -7 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

The image of $\begin{bmatrix} 5 \\ -2 \\ -7 \end{bmatrix}$ is $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$ and $\underline{\underline{a}}$ preimage of $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} 5 \\ -2 \\ -7 \end{bmatrix}$.

The preimages of $\vec{0}$ all come from the set $\left\{ \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$

Linear Transformations

A ***linear transformation***, $T:\mathbb{R}^n \to \mathbb{R}^m$, is a transformation that satisfies both of the following properties.

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \text{ and } T(c\vec{u}) = cT(\vec{u}) \;\; \forall \vec{u},\vec{v} \in \mathbb{R}^n \text{ and } c \in \mathbb{R}$$

Example

Show that $T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$ is a linear transformation whereas $T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 1 \end{bmatrix}$ is not.

$$\begin{aligned} T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) &= T_1\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right) \\ &= \begin{bmatrix} x_2+y_2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} y_2 \\ 0 \end{bmatrix} \\ &= T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + T_1\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) \end{aligned}$$

$$\begin{aligned} T_1\left(c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) &= T_1\left(\begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}\right) \\ &= \begin{bmatrix} cx_2 \\ 0 \end{bmatrix} \\ &= c\begin{bmatrix} x_2 \\ 0 \end{bmatrix} \\ &= c\,T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) \end{aligned}$$

QED

To show that T_2 is <u>not</u> a linear transformation we only need show one counterexample.

$$T_2\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T_2\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T_2\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) + T_2\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

QED

$\begin{bmatrix}-2\\3\end{bmatrix} = -2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}$

$= -2\vec{e}_1 + 3\vec{e}_2$

Example

Draw $T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right)$ given that T is a linear transformation and the images for $\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ are those shown in Figure 1.

$T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right) = T(-2\vec{e}_1 + 3\vec{e}_2)$

$= T(-2\vec{e}_1) + T(3\vec{e}_2)$

$= -2T(\vec{e}_1) + 3T(\vec{e}_2)$

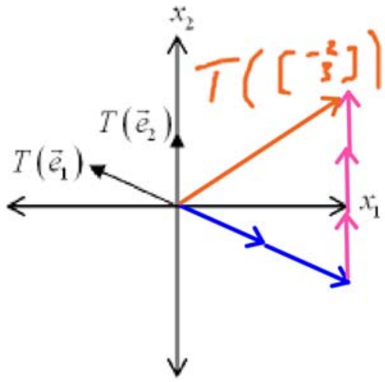


Figure 1: Transformation Vectors

A definition and a very convenient fact

The identity matrix, I_n, is the n x n matrix that has 1s for every entry along the main diagonal and 0 for every other entry.

If we let $\vec{e}_i$ represent the i^{th} column of I_n, then the images of $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ under the linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ completely determines **all** of the images under T.

Example

image is 2-d

Suppose that $T:\mathbb{R}^3 \to \mathbb{R}^2$ and that $T(\vec{e}_1) = \begin{bmatrix}3\\-2\end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix}-1\\4\end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix}5\\0\end{bmatrix}$.

preimage is 3-d

$\begin{bmatrix}1\\0\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\\0\end{bmatrix}$ $\begin{bmatrix}0\\0\\1\end{bmatrix}$

a. Determine $T\left(\begin{bmatrix}-6 & 2 & 1\end{bmatrix}^{\mathrm{T}}\right)$.

$T\left(\begin{bmatrix}-6\\2\\1\end{bmatrix}\right) = T(-6\vec{e}_1 + 2\vec{e}_2 + \vec{e}_3)$

$= -6T(\vec{e}_1) + 2T(\vec{e}_2) + T(\vec{e}_3)$

$= -6\begin{bmatrix}3\\-2\end{bmatrix} + 2\begin{bmatrix}-1\\4\end{bmatrix} + \begin{bmatrix}5\\0\end{bmatrix}$

$= \begin{bmatrix}-15\\20\end{bmatrix}$

$T(\vec{e}_1)=\begin{bmatrix}3\\-2\end{bmatrix}$, $T(\vec{e}_2)=\begin{bmatrix}-1\\4\end{bmatrix}$, and $T(\vec{e}_3)=\begin{bmatrix}5\\0\end{bmatrix}$

b. Find a matrix, M, with the property that $T(\vec{x})=M\vec{x}\ \ \forall\ \ \vec{x}\in\mathbb{R}^3$.

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3)$$
$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + x_3 T(\vec{e}_3)$$
$$= x_1\begin{bmatrix}3\\-2\end{bmatrix} + x_2\begin{bmatrix}-1\\4\end{bmatrix} + x_3\begin{bmatrix}5\\0\end{bmatrix}$$
$$= \begin{bmatrix}3x_1 & -x_2 & +5x_3\\ -2x_1 & +4x_2 & +0x_3\end{bmatrix}$$
$$= \begin{bmatrix}3 & -1 & 5\\ -2 & 4 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_2\end{bmatrix}$$

Theorem

Every transformation of form $T(\vec{x})=A\vec{x}$ is a linear transformation and if T is a linear transformation there exists a unique matrix A such that $T(\vec{x})=A\vec{x}$.

Example

Suppose that $T:\mathbb{R}^2\to\mathbb{R}^2$ is a linear transformation. Find the matrix for T if $T(\vec{e}_1)=\begin{bmatrix}a_{11}\\a_{21}\end{bmatrix}$ and $T(\vec{e}_2)=\begin{bmatrix}a_{12}\\a_{22}\end{bmatrix}$.

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$
$$= x_1\begin{bmatrix}a_{11}\\a_{21}\end{bmatrix} + x_2\begin{bmatrix}a_{12}\\a_{22}\end{bmatrix}$$
$$= \begin{bmatrix}a_{11}x_1 + a_{12}x_2\\ a_{21}x_1 + a_{22}x_2\end{bmatrix}$$
$$= \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

AB

$m\times p$ $\quad p\times n$ $\quad$ product $\quad m\times n$

Example

Show that $T(\vec{x}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\vec{x}$ is a linear transformation.

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) = T\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right)$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a+c \\ b+d \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(a+c) + a_{12}(b+d) \\ a_{21}(a+c) + a_{22}(b+d) \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}a + a_{12}b) + (a_{11}c + a_{12}d) \\ (a_{21}a + a_{22}b) + (a_{21}c + a_{22}d) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}$$

$$= T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right)$$

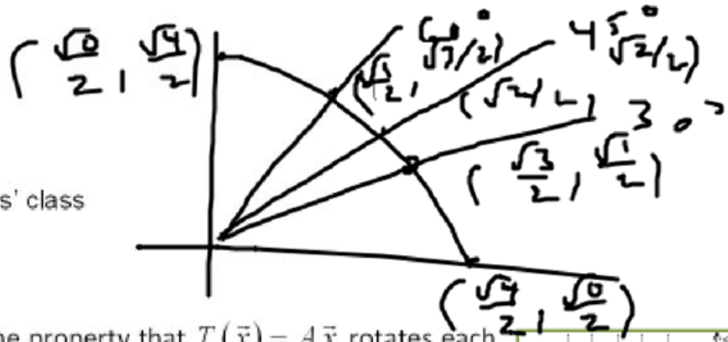
$$T\left(k\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} ka \\ kb \end{bmatrix}$$

$$= \begin{bmatrix} ka_{11}a + ka_{12}b \\ ka_{21}a + ka_{22}b \end{bmatrix}$$

$$= k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

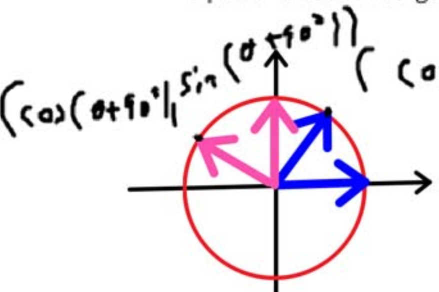
$$= kT\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$$

QED



Example

Find a matrix A with the property that $T(\vec{x}) = A\vec{x}$ rotates each vector in the x_1x_2-plane by 60° in the counter-clockwise direction. Illustrate the effect of the transformation on the "unit square" shown in Figure 2.



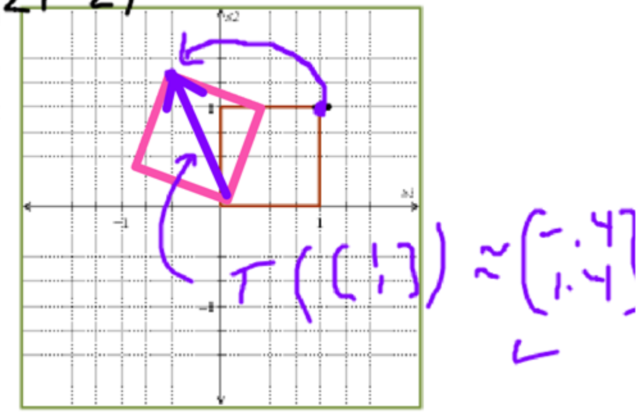


Figure 2: Rotated "unit square"

$$T(\vec{e}_1) = \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$$

$$T(\vec{e}_2) = \begin{bmatrix}\cos(\theta+90°)\\ \sin(\theta+90°)\end{bmatrix}$$

$$= \begin{bmatrix}\cos(\theta)\cos(90°) - \sin(\theta)\sin(90°)\\ \sin(\theta)\cos(90°) + \cos(\theta)\sin(90°)\end{bmatrix}$$

$$= \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}$$

VOILA! The transformation matrix for a rotation of θ is $A = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$

For $\theta = 60°$

$$A = \begin{bmatrix}\cos(60°) & -\sin(60°)\\ \sin(60°) & \cos(60°)\end{bmatrix} = \begin{bmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1/2\\ \sqrt{3}/2\end{bmatrix} \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sqrt{3}/2\\ 1/2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = A\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = A\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \begin{bmatrix}1/2 - \frac{\sqrt{3}}{2}\\ \sqrt{3}/2 + \frac{1}{2}\end{bmatrix}$$

$$\approx \begin{bmatrix}-.4\\ 1.4\end{bmatrix}$$

A transformation that is both surjective and injective is called bijective

Definitions and a Theorem

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** $\mathbb{R}^m$ if and only if the range of the transformation is $\mathbb{R}^m$; that is, the transformation is onto $\mathbb{R}^m$ if and only if every vector in $\mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$.

Surjective

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if and only if $T(\vec{u}) = T(\vec{v}) \Leftrightarrow \vec{u} = \vec{v}$. It is trivially shown that the transformation is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{0}$.

Injective

Example

Let $A = \begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix}$. Determine whether or not the linear transformation $T(\vec{x}) = A\vec{x}$ is onto and also whether or not it is one-to-one.

T maps $\mathbb{R}^3$ to $\mathbb{R}^2$ so T cannot possibly be one-to-one (injective).

To answer the "onto" question we need to determine if $\begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ can ever have no solutions.

$$\left[\begin{array}{ccc|c} 3 & -2 & -16 & b_1 \\ 2 & 4 & 16 & b_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & * \\ 0 & 1 & 5 & * \end{array}\right]$$

The RREF matrix never results in $0 = \text{not } 0$, so the system has solutions regardless of the specific vector $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

The transformation is onto $\mathbb{R}^2$