Transformations

A ***transformation***, T, from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function that assigns to each vector in $\mathbb{R}^n$ a unique vector in $\mathbb{R}^m$. If $T(\vec{x}) = \vec{b}$, we say that $\vec{b}$ is the ***image*** of $\vec{x}$ under T.

$\mathbb{R}^n$ is called the ***domain*** of T and $\mathbb{R}^m$ is called the ***codomain*** of T. The set of all images found under T is called the ***range*** of T.

Example

Suppose that T is the transformation defined by the rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$. What are the domain, codomain, and range of T? What is the image of $\vec{x}$ where $\vec{x} = \begin{bmatrix} 5 & -2 & -7 \end{bmatrix}^{\mathrm{T}}$? Describe the set of vectors whose images are $\vec{0}$.

The domain of T is $\mathbb{R}^3$, the codomain is $\mathbb{R}^2$, the range is $\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$

$T\left(\begin{bmatrix} 5 \\ -2 \\ -7 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$; the image of $\begin{bmatrix} 5 \\ -2 \\ -7 \end{bmatrix}$ under T is $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 0$

$\therefore$ The set of <u>preimages</u> of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is:

$$\left\{ \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$$

$f(x) = x^2$

Linear Transformations

A ***linear transformation***, $T:\mathbb{R}^n \to \mathbb{R}^m$, is a transformation that satisfies both of the following properties.

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \text{ and } T(c\vec{u}) = cT(\vec{u}) \ \ \forall \vec{u},\vec{v} \in \mathbb{R}^n \text{ and } c \in \mathbb{R}$$

Example

Show that $T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$ is a linear transformation whereas $T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 1 \end{bmatrix}$ is not.

$$T_1\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = T_1\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right)$$
$$= \begin{bmatrix} u_2+v_2 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} v_2 \\ 0 \end{bmatrix}$$
$$= T_1\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) + T_1\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right)$$

$$T_1\left(c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = T_1\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}\right)$$
$$= \begin{bmatrix} cu_2 \\ 0 \end{bmatrix}$$
$$= c\begin{bmatrix} u_2 \\ 0 \end{bmatrix}$$
$$= c\,T_1\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right)$$

$\therefore$ T_1 is a linear transformation

$$T_2\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T_2\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = T_2\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) + T_2\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

QED

Example

Draw $T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right)$ given that the images for $\vec{e}_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{e}_2=\begin{bmatrix}0\\1\end{bmatrix}$ are those shown in Figure 1 and that T is a linear transformation.

$$T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right)=T\left(-2\vec{e}_1+3\vec{e}_2\right)$$
$$=T(-2\vec{e}_1)+T(3\vec{e}_2)$$
$$=-2T(\vec{e}_1)+3T(\vec{e}_2)$$

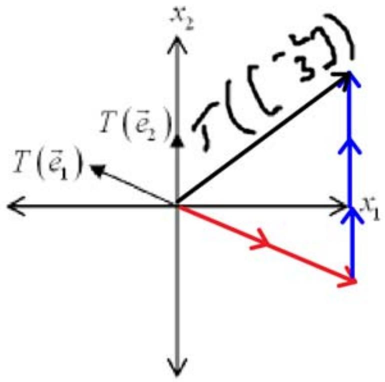


Figure 1: Transformation Vectors

A definition and a very convenient fact

The identity matrix, I_n, is the n x n matrix that has 1s for every entry along the main diagonal and 0 for every other entry.

If we let $\vec{e}_i$ represent the i^{th} column of I_n, then the images of $\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_n$ under the linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$ completely determines **all** of the images under T.

Example

Suppose that $T:\mathbb{R}^3\to\mathbb{R}^2$ and that $T(\vec{e}_1)=\begin{bmatrix}3\\-2\end{bmatrix}$, $T(\vec{e}_2)=\begin{bmatrix}-1\\4\end{bmatrix}$, and $T(\vec{e}_3)=\begin{bmatrix}5\\0\end{bmatrix}$.

a. Determine $T\left(\begin{bmatrix}-6 & 2 & 1\end{bmatrix}^{\mathrm{T}}\right)$.

$$T\left(\begin{bmatrix}-6\\2\\1\end{bmatrix}\right)=T\left(-6\vec{e}_1+2\vec{e}_2+\vec{e}_3\right)$$
$$=T(-6\vec{e}_1)+T(2\vec{e}_2)+T(\vec{e}_3)$$
$$=-6T(\vec{e}_1)+2T(\vec{e}_2)+T(\vec{e}_3)$$
$$=-6\begin{bmatrix}3\\-2\end{bmatrix}+2\begin{bmatrix}-1\\4\end{bmatrix}+\begin{bmatrix}5\\0\end{bmatrix}$$
$$=\begin{bmatrix}-15\\20\end{bmatrix}$$

$T(\vec{e}_1)=\begin{bmatrix}3\\-2\end{bmatrix}$, $T(\vec{e}_2)=\begin{bmatrix}-1\\4\end{bmatrix}$, and $T(\vec{e}_3)=\begin{bmatrix}5\\0\end{bmatrix}$

b. Find a matrix, M, with the property that $T(\vec{x})=M\,\vec{x}\;\;\forall\;\;\vec{x}\in\mathbb{R}^3$.

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=T\left(x_1\vec{e}_1+x_2\vec{e}_2+x_3\vec{e}_3\right)$$
$$=x_1T(\vec{e}_1)+x_2T(\vec{e}_2)+x_3T(\vec{e}_3)$$
$$=x_1\begin{bmatrix}3\\-2\end{bmatrix}+x_2\begin{bmatrix}-1\\4\end{bmatrix}+x_3\begin{bmatrix}5\\0\end{bmatrix}$$
$$=\begin{bmatrix}3&-1&5\\-2&4&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Theorem

Every transformation of form $T(\vec{x})=A\vec{x}$ is a linear transformation and if T is a linear transformation there exists a unique matrix A such that $T(\vec{x})=A\vec{x}$.

Example

Suppose that $T:\mathbb{R}^2\to\mathbb{R}^2$ is a linear transformation. Find the matrix for T.

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=T(x_1\vec{e}_1+x_2\vec{e}_2)$$
$$=x_1T(\vec{e}_1)+x_2T(\vec{e}_2)$$
$$=x_1\begin{bmatrix}a_1\\a_2\end{bmatrix}+x_2\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

where $T(\vec{e}_1)=\begin{bmatrix}a_1\\a_2\end{bmatrix}$ + $T(\vec{e}_2)=\begin{bmatrix}b_1\\b_2\end{bmatrix}$

$$=\begin{bmatrix}a_1x_1+b_1x_2\\a_2x_1+b_2x_2\end{bmatrix}$$
$$=\begin{bmatrix}a_1&b_1\\a_2&b_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$\therefore\; T(\vec{x})=\begin{bmatrix}a_1&b_1\\a_2&b_2\end{bmatrix}\vec{x}$ where $T(\vec{e}_1)=\begin{bmatrix}a_1\\a_2\end{bmatrix}$ and $T(\vec{e}_2)=\begin{bmatrix}b_1\\b_2\end{bmatrix}$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\approx \begin{bmatrix}-.4\\1.3\end{bmatrix}$$

Example

Show that $T(\vec{x}) = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\vec{x}$ is a linear transformation.

$$T(\vec{u}+\vec{v}) = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}u_1+v_1\\ u_2+v_2\end{bmatrix}$$

$$= \begin{bmatrix}a_{11}(u_1+v_1)+a_{12}(u_2+v_2)\\ a_{21}(u_1+v_1)+a_{22}(u_2+v_2)\end{bmatrix}$$

$$= \begin{bmatrix}a_{11}u_1+a_{12}u_2\\ a_{21}u_1+a_{22}u_2\end{bmatrix} + \begin{bmatrix}a_{11}v_1+a_{12}v_2\\ a_{21}v_1+a_{22}v_2\end{bmatrix}$$

$$= \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} + \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

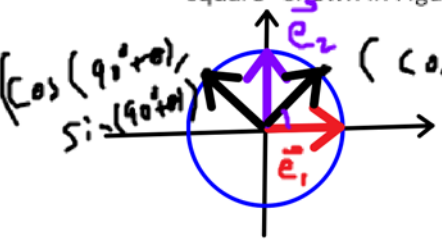
$$= T(\vec{u}) + T(\vec{v})$$

$$T(c\vec{u}) = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}cu_1\\cu_2\end{bmatrix} = \begin{bmatrix}ca_{11}u_1+ca_{12}u_2\\ ca_{21}u_1+ca_{22}u_2\end{bmatrix} = c\begin{bmatrix}a_{11}u_1+a_{12}u_2\\ a_{21}u_1+a_{22}u_2\end{bmatrix} = cT(\vec{u})$$

QED

Example

Find a matrix A with the property that $T(\vec{x}) = A\vec{x}$ rotates each vector in the x_1x_2-plane by 60° in the counter-clockwise direction. Illustrate the effect of the transformation on the "unit square" shown in Figure 2.



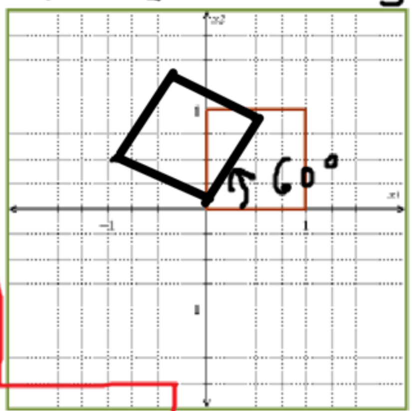


Figure 2: Rotated "unit square"

The unit circle

Call the rotation angle θ.

The θ-rotation matrix $\mathbb{R}^2 \to \mathbb{R}^2$ is

$$M = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$$

$$T(\vec{e}_1) = \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$$

$$T(\vec{e}_2) = \begin{bmatrix}\cos(\theta+90°)\\ \sin(\theta+90°)\end{bmatrix}$$

$$= \begin{bmatrix}\cos(\theta)\cos(90°) - \sin(\theta)\sin(90°)\\ \sin(\theta)\cos(90°) + \cos(\theta)\sin(90°)\end{bmatrix}$$

$$= \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}$$

For $60°$-counterclockwise rotation

$$T(\vec{x}) = \begin{bmatrix}\cos(60°) & -\sin(60°)\\ \sin(60°) & \cos(60°)\end{bmatrix}\vec{x}$$

$$= \begin{bmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix}\vec{x}$$

$$T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix} \qquad T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1/2\\ \sqrt{3}/2\end{bmatrix} \approx \begin{bmatrix}.5\\.9\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sqrt{3}/2\\ 1/2\end{bmatrix} \approx \begin{bmatrix}-.9\\.5\end{bmatrix}$$

bijective
Both

Definitions and a Theorem

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** $\mathbb{R}^m$ if and only if the range of the transformation is $\mathbb{R}^m$; that is, the transformation is onto $\mathbb{R}^m$ if and only if every vector in $\mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$. (surjective)

The linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if and only if $T(\vec{u}) = T(\vec{v}) \Leftrightarrow \vec{u} = \vec{v}$. It is trivially shown that the transformation is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{0}$.
(injective)

Example

Let $A = \begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix}$. Determine whether or not the linear transformation $T(\vec{x}) = A\vec{x}$ is onto and also whether or not it is one-to-one.

$$T(\vec{x}) = \begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix}\vec{x}$$

$$= \begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 3x_1 - 2x_2 - 16x_3 \\ 2x_1 + 4x_2 + 16x_3 \end{bmatrix}$$

$T: \mathbb{R}^3 \to \mathbb{R}^2$ which means that T cannot <u>Possibly</u> <u>be</u> one-to-one.

If T is onto $\mathbb{R}^2$, then

$$\begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$ will always have a solution regardless of the values of b_1 & b_2.

$$\begin{bmatrix} 3 & -2 & -16 \\ 2 & 4 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \end{bmatrix}$$ so the rref of

$$\left[\begin{array}{ccc|c} 3 & -2 & -16 & b_1 \\ 2 & 4 & 16 & b_2 \end{array}\right]$$ will never have a string of zero to the left of the augment line

$\therefore$ The system always has a solution and, consequently, T is onto $\mathbb{R}^2$.